ENDPOINT ESTIMATES FOR RUBIO DE FRANCIA OPERATORS

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Abstract. The extrapolation theory of Rubio de Francia provides a tool to obtain A_p weighted estimates on L^p spaces for every $1 < p < \infty$, starting from information at a single $1 < p_0 < \infty$. However, the endpoint case $p = 1$ cannot be reached in general. Classical extrapolation arguments in the sense of Yano can be added to this setting to deduce results close to L^1 without weights. In this paper, we present different approaches that produce endpoint estimates with respect to the whole A_1 class. We give applications to the Carleson operator and maximally modulated singular integrals among others.

1. Introduction and motivation

In 1984, J. L. Rubio de Francia [38] presented one of the most important extrapolation results in the context of weighted L^p spaces. Recall that, given a measure space (Ω, μ) and $1 \leq p < \infty$, we define $L^p(\mu)$ as the set of μ-measurable functions satisfying

$$\|f\|_{L^p(\mu)} = \left(\int_\Omega |f(x)|^p d\mu(x) \right)^{1/p} < \infty.$$

When $\Omega = \mathbb{R}^n$ and the measure $d\mu = w dx$ is given by a non-negative, locally integrable function w (that we call weight), then we write $L^p(w)$. Rubio de Francia's extrapolation theorem can be stated as follows:

Theorem 1.1 (Rubio de Francia, 1984)**.** *Given a sublinear operator T, if for some $1 \leq p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded, then, for every $1 < p < \infty$ and every $w \in A_p$,

$$T : L^p(w) \longrightarrow L^p(w)$$

is also bounded.

The classes of weights appearing in Theorem 1.1 are the so-called A_p weights, introduced by B. Muckenhoupt [34] in 1972. For $1 < p < \infty$, we say that $w \in A_p$ whenever

$$\|w\|_{A_p} = \sup_Q \frac{w(Q)}{|Q|} \left(\frac{w^{1-p'}(Q)}{|Q|} \right)^{p-1} < \infty,$$

and $u \in A_1$ if

$$Mu(x) \leq Cu(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

2010 *Mathematics Subject Classification.* 42B99, 46E30.
Key words and phrases. Rubio de Francia operators, weighted estimates, A_p weights, Yano's extrapolation.
The authors were supported by grants MTM2013-40985-P and 2014SGR289.

with $\|u\|_{A_1}$ being the least constant $C \geq 1$ that can be taken in such an inequality. The operator M involved is the Hardy-Littlewood maximal operator, defined on locally integrable functions f by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x. In fact, as shown by Muckenhoupt in the same paper, these A_p classes characterize the L^p boundedness of M. More precisely, when $1 < p < \infty$,

$$M : L^p(w) \longrightarrow L^p(w)$$

is bounded if and only if $w \in A_p$, and

$$M : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded if and only if $u \in A_1$.

Since 1984, Theorem 1.1 has been improved in different directions (see [16, 17, 22] for further details). For instance, it can be shown that the result is still true for general operators (not necessarily sublinear), or if we change the strong-type estimates by weak-type ones. Also, the following quantitative version of Theorem 1.1 was given by Dragičević, Grafakos, Pereyra and Petermichl in [20] (see also [22]):

Theorem 1.2 (DGPP, 2005). *Given an operator T, if for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, with φ an increasing function on $(0, \infty)$, then, for every $1 < p < p_0$ and every $w \in A_p$,

$$T : L^p(w) \longrightarrow L^p(w)$$

is bounded with constant controlled by

$$C_1 \varphi \left(\frac{C_2 \|w\|_{A_p}^{\frac{p_0 - 1}{p - 1}}}{(p-1)^{p_0 - 1}} \right), \qquad \text{as } p \to 1^+. \tag{1.1}$$

Here C_1 and C_2 are two positive constants independent of p and w.

At this point we should emphasize that, under the hypotheses of Theorem 1.2, it is not possible to extrapolate down to $p = 1$, not even if we are only seeking a weak-type estimate without weights. To illustrate this limitation, we can consider the composition $M^2 = M \circ M$. Clearly, $M^2 : L^p(w) \to L^p(w)$ for every $1 < p < \infty$ and $w \in A_p$, but $M^2 : L^1(\mathbb{R}^n) \nrightarrow L^{1,\infty}(\mathbb{R}^n)$. Therefore, finding enpoint estimates (close to L^1) for operators under the assumptions of Rubio de Francia's Theorem 1.2 becomes an interesting goal.

Now, looking at Theorem 1.2, one can immediately see that, if T satisfies its hypotheses with $\varphi(t) = t^s$ for some $s > 0$, then

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \qquad \|T\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq \frac{C}{(p-1)^{s(p_0 - 1)}}. \tag{1.2}$$

The problem of approaching the endpoint L^1 only from information in L^p for $p > 1$ such as the one obtained in (1.2) is the starting point of another extrapolation theory, initiated by S. Yano in 1951. His original result [42] can be stated as follows:

Theorem 1.3 (Yano, 1951)**.** *Fix (Ω, μ) a finite measure space, $p_0 > 1$ and $m > 0$. If T is a sublinear operator such that, for every $1 < p \le p_0$,*

$$T : L^p(\mu) \longrightarrow L^p(\mu)$$

is bounded with norm essentially controlled by $(p-1)^{-m}$, then,

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\mu)$$

is bounded.

Recall that $L(\log L)^m(\mu) \subsetneq L^1(\mu)$ is the space of μ-measurable functions such that

$$\|f\|_{L(\log L)^m(\mu)} = \int_0^\infty f^*_\mu(t)\left(1 + \log_+\left(\frac{1}{t}\right)\right)^m dt < \infty,$$

where, as usual, $\log_+$ denotes the positive part of the logarithm and f^*_μ is the decreasing rearrangement of f with respect to μ defined by

$$f^*_\mu(t) = \inf\{y > 0 : \lambda^\mu_f(y) \le t\}, \qquad \lambda^\mu_f(t) := \mu(\{x \in \Omega : |f(x)| > t\}).$$

Theorem 1.3 can be extended to σ-finite measures and, also, improved in order to have weaker hypotheses and a better range space. See [7, 8] for more details on this extension. Before we make its statement precise, let us recall that, given $1 \le p < \infty$ and $0 < q < \infty$, the Lorentz spaces $L^{p,q}(\mu)$ are defined as the set of μ-measurable functions such that

$$\|f\|_{L^{p,q}(\mu)} = \left(p\int_0^\infty (t\lambda^\mu_f(t)^{1/p})^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty (t^{1/p} f^*_\mu(t))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{t>0} t\lambda^\mu_f(t)^{1/p} = \sup_{t>0} t^{1/p} f^*_\mu(t).$$

Notice that if $p = q$, then $L^{p,p}(\mu) = L^p(\mu)$, and for every $1 \le p < \infty$, we have the following chain of continuous inclusions:

$$L^{p,q_1}(\mu) \subseteq L^{p,q_2}(\mu) \subseteq L^{p,\infty}(\mu), \quad 0 < q_1 < q_2 < \infty.$$

We will also need to consider general log-type spaces. For simplicity, we will adopt the following notation:

$$\log_1(x) = 1 + \log_+(x) \quad \text{and} \quad \log_k(x) = \log_1 \log_{k-1}(x), \text{ for } k > 1.$$

For natural numbers $1 \le j_1 < j_2 < \cdots < j_n$ and positive real numbers $m_1, ..., m_n > 0$, we define the space

$$L(\log_{j_1} L)^{m_1} \cdots (\log_{j_n} L)^{m_n}(\mu)$$

as the set of μ-measurable functions such that

$$\|f\|_{L(\log_{j_1} L)^{m_1}\cdots(\log_{j_n} L)^{m_n}(\mu)} = \int_0^\infty f^*_\mu(t) \log_{j_1}^{m_1}\left(\frac{1}{t}\right) \cdots \log_{j_n}^{m_n}\left(\frac{1}{t}\right) dt < \infty.$$

This is the modern version of Theorem 1.3 that we will use:

Theorem 1.4 ([7, 8]). *Given a σ-finite measure μ and $m > 0$, if a sublinear operator T satisfies that*

$$T : L^{p,1}(\mu) \longrightarrow L^p(\mu)$$

is bounded with constant less than or equal to $(p-1)^{-m}$ for every $1 < p \le p_0$, then

$$T : L(\log L)^m(\mu) \longrightarrow E_m(\mu)$$

is also bounded where $E_m(\mu)$ is the space of μ-measurable functions such that

$$\|f\|_{E_m(\mu)} = \sup_{t>0} \frac{t f^{**}_\mu(t)}{\log_1^m t} < \infty,$$

and $f^{**}_\mu(t) := \frac{1}{t}\int_0^t f^*_\mu(s)ds$.

Remark 1.5. *Let us just emphasize that the constant of the operator T on $L(\log L)^m(\mu)$ in the previous theorem may depend on p_0 but not on μ or T. This is the standard situation in all Yano type results that will appear in the paper (see Theorems 2.6 and 2.16).*

Therefore, as a consequence of (1.2) and Theorem 1.4, we can immediately deduce the following endpoint result for Rubio de Francia operators:

Corollary 1.6. *Let $1 < p_0 < \infty$, $s > 0$ and let T be a sublinear operator such that*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $\|w\|^s_{A_{p_0}}$. Then,

$$T : L(\log L)^{s(p_0-1)}(\mathbb{R}^n) \longrightarrow E_{s(p_0-1)}(\mathbb{R}^n)$$

is also bounded.

Estimates of the type (1.2) (or similar ones when $p \to \infty$) for operators satisfying the hypotheses of Theorem 1.2 with $\varphi(t) = t^s$ have appeared in different situations in the literature (see, for example, [19, 23, 33]). In particular, in [33], a converse argument can be used to find optimal values of s.

However, as we shall see in this paper, there is still much more to say. Our main goal is to obtain endpoint results close to $L^1(u)$ for Rubio de Francia operators with respect to A_1 weights. To this end, we will combine several variants of both Rubio de Francia and Yano extrapolation theories, and then study under which Rubio de Francia condition on T we can obtain the best endpoint estimate.

Before going on, and following the suggestion of the referee, we include a list of the properties of A_p weights that are going to be important for our purposes. Most of them can be found in Chapter 7 of [21].

i) Coifman-Rochberg characterization of A_1 weights: Every weight u in the class A_1 is of the form $u(x) = k(x)(Mf)^\delta$ where $k, k^{-1} \in L^\infty$, $f \in L^1_{loc}$ and $0 \le \delta < 1$. Moreover, if the funcion $k(x) = 1$, then

$$\|(Mf)^\delta\|_{A_1} \le \frac{C}{1-\delta}, \tag{1.3}$$

with C some universal constant independent of δ and f.

ii) P. Jones factorization theorem: A weight w is in the class A_p if and only if there exist two weights u_0 and u_1 in A_1 so that $w = u_0^{1-p} u_1$. Moreover,

$$\|u_0^{1-p} u_1\|_{A_p} \leq \|u_0\|_{A_1}^{p-1} \|u_1\|_{A_1}. \tag{1.4}$$

iii) Clearly, if $u \in A_1$, then $u \in A_p$ and $\|u\|_{A_p} \leq \|u\|_{A_1}$.
iv) By Hölder's inequality, if $u \in A_1$ and $\delta < 1$, then $u^\delta \in A_1$ and

$$\|u^\delta\|_{A_1} \leq \|u\|_{A_1}^\delta. \tag{1.5}$$

v) If $u_0, u_1 \in A_1$ and $1 < p < p_0$, then $u_0^{p-p_0} u_1 \in A_{p_0}$ and

$$\|u_0^{p-p_0} u_1\|_{A_{p_0}} \leq \|u_0\|_{A_1}^{p_0-1} \|u_1\|_{A_1}. \tag{1.6}$$

To see this, we observe that

$$u_0^{p-p_0} u_1 = \left(u_0^{\frac{p_0-p}{p_0-1}}\right)^{1-p_0} u_1.$$

Hence, by (1.4) and (1.5) and since $\|u\|_{A_1} \geq 1$,

$$\|u_0^{p-p_0} u_1\|_{A_{p_0}} \leq \|u_0^{\frac{p_0-p}{p_0-1}}\|_{A_1}^{p_0-1} \|u_1\|_{A_1} \leq \|u_0\|_{A_1}^{p_0-p} \|u_1\|_{A_1} \leq \|u_0\|_{A_1}^{p_0-1} \|u_1\|_{A_1}.$$

The paper is organized as follows: in Section 2 we present our main results for operators under similar assumptions to those in Rubio de Francia's theorem. Then we will see how different types of operator norms associated with them blow up as p tends to 1, and how this information can be exploited to extrapolate in the sense of Yano. Next, in Section 3 we improve a particular endpoint estimate which is related to the extrapolation theory introduced in [10]. Finally, in Section 4, we see how our results can be useful in different applications.

From now on, we will write $x \lesssim y$ when there is a positive constant $C > 0$ such that $x \leq Cy$. If both $x \lesssim y$ and $y \lesssim x$, then we write $x \approx y$. The constants involved do not depend on any parameter that is fixed in its context. Moreover, throughout the paper constants such as $C, C_1, C_2, \ldots$ will denote universal constants that may depend only on the fixed parameters (such as p_0 or μ). In the case $d\mu = u(x)dx$, with $u \in A_1$ fixed, we shall indicate the behaviour of the constant in u whenever possible. Otherwise, we shall write C_u.

Finally, we want to thank the referee for his/her thorough report and for all the comments and remarks that have improved the final version of this paper.

2. Endpoint estimates for Rubio de Francia operators

Given a sublinear operator T defined on the set of μ-measurable functions, we say that T is of strong-type (p_0, p_0) if

$$T : L^{p_0}(\mu) \longrightarrow L^{p_0}(\mu).$$

It is of weak-type (p_0, p_0) if

$$T : L^{p_0}(\mu) \longrightarrow L^{p_0,\infty}(\mu).$$

It is of restricted weak-type (p_0, p_0) if

$$T : L^{p_0,1}(\mu) \longrightarrow L^{p_0,\infty}(\mu),$$

and finally, of strong weak-type (p_0, p_0) if

$$T : L^{p_0,\infty}(\mu) \longrightarrow L^{p_0,\infty}(\mu).$$

In the top part of Table 1, we summarize the four basic cases that we will study in this section. Notice that the assumptions (first row) will be some of the previous boundedness at level $p_0 > 1$ with respect to A_{p_0} weights, and with constant essentially controlled by a power of $\|w\|_{A_{p_0}}$. From here, we will deduce the same boundedness but for every $1 < p < p_0$ with respect to a fixed weight in A_1, keeping track of how the constant blows up when $p \to 1$ (second row). This information will then be used to extrapolate in the sense of Yano with a suitable technique, and reach an endpoint space of logarithmic type with A_1 weights (third row). In the bottom part of the table, we also show what we obtain if we go from the hypotheses of Subsection 2.3 to those in Subsection 2.4, and then extrapolate from there.

Subsection 2.1	**Subsection 2.2**	**Subsection 2.3**	**Subsection 2.4**
$T : L^{p_0}(w) \to L^{p_0}(w)$ $\|T\| \lesssim \|w\|^s_{A_{p_0}}$	$T : L^{p_0}(w) \to L^{p_0,\infty}(w)$ $\|T\| \lesssim \|w\|^\sigma_{A_{p_0}}$	$T : L^{p_0,1}(w) \to L^{p_0,\infty}(w)$ $\|T\| \lesssim \|w\|^r_{A_{p_0}}$	$T : L^{p_0,\infty}(w) \to L^{p_0,\infty}(w)$ $\|T\| \lesssim \|w\|^\alpha_{A_{p_0}}$
$T : L^p(u) \to L^p(u)$ $\|T\| \lesssim \frac{C_u}{(p-1)^{s(p_0-1)}}$	$T : L^p(u) \to L^{p,\infty}(u)$ $\|T\| \lesssim \frac{C_u}{(p-1)^{\sigma(p_0-1)}}$	$T : L^{p,p/p_0}(u) \to L^{p,\infty}(u)$ $\|T\| \lesssim \frac{C_u}{(p-1)^{r(p_0-1)}}$	$T : L^{p,\infty}(u) \to L^{p,\infty}(u)$ $\|T\| \lesssim \frac{C_u}{(p-1)^{\alpha(p_0-1)}}$
$L(\log L)^{s(p_0-1)}(u)$	$L(\log L)^{\sigma(p_0-1)} \log_3 L(u)$	$L(\log L)^{r(p_0-1)+b} \log_3 L(u)$ $b = \min(1, p_0 - 1)$ (ε, δ)-atomic case: $b = 0$	$[L(\log L)^{\alpha(p_0-1)-1} \log_3 L(u)]_1$

Subsection 2.5
$T : L^{p_0,1}(w) \to L^{p_0,\infty}(w)$ $\|T\| \lesssim \|w\|^r_{A_{p_0}}$
$T : L^{p_0,\infty}(w) \to L^{p_0,\infty}(w)$ $\|T\| \lesssim \|w\|^{r+p_0'-1}_{A_{p_0}}$
$[L(\log L)^{r(p_0-1)} \log_3 L(u)]_1$

Table 1. In all the cases, $w \in A_{p_0}$, $u \in A_1$ and $1 < p < p_0$.

For later purposes, let us mention that, using that

$$M : L^1(u) \to L^{1,\infty}(u), \qquad \|M\|_{L^1(u)\to L^{1,\infty}(u)} \lesssim \|u\|_{A_1},$$

and that M is bounded on L^∞ with constant 1, one can easily prove that, for every $u \in A_1$,

$$(Mf)^*_u(t) \lesssim \frac{\|u\|_{A_1}}{t} \int_0^t f^*_u(s)ds.$$

From here, it follows that

$$\|M\|_{L^p(u)\to L^p(u)} \lesssim \frac{\|u\|_{A_1}}{p-1}, \qquad \|M\|_{L^{p,\infty}(u)\to L^{p,\infty}(u)} \lesssim \frac{\|u\|_{A_1}}{p-1}. \tag{2.1}$$

Also, the sharp weak-type (p,p) estimate for M due to S. M. Buckley [6] states that

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim \|w\|_{A_p}^{1/p}. \tag{2.2}$$

2.1. **From strong-type to strong-type.** In this subsection, our starting hypothesis will be the classical one in Rubio de Francia's extrapolation theory. That is, for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$, T is of strong type (p_0, p_0) with respect to the weight w, with constant $\varphi(\|w\|_{A_{p_0}})$.

From (1.1), we can see that, even if $\varphi(t) = t^s$ for some $s > 0$, the blow-up of the constant is exponential except in the unweighted case, and thus Yano's theory cannot be applied directly. To avoid this problem, our first result is an easy modification of the proof of Theorem 1.2 in [22]. We believe it is well-known to the experts in the topic but we have not found a proof in the literature and hence we include it for the sake of completeness.

Theorem 2.1. *Given an operator T, if for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, where $\varphi : [0,\infty) \to [0,\infty)$ is an increasing function, then, given $u \in A_1$,

$$T : L^p(u) \longrightarrow L^p(u)$$

is bounded for every $1 < p < p_0$ with constant controlled by

$$C_1 \min\left[\varphi\left(\frac{C_2\|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}}\right),\ \left(\frac{\|u\|_{A_1}}{p-1}\right)^{1-\frac{p}{p_0}} \varphi\left(\frac{C_3\|u\|_{A_1}}{(p-1)^{p_0-1}}\right)\right]. \tag{2.3}$$

Proof. In order to get the behavior of the constant, we shall proceed in two different ways: 1) Given $u \in A_1$, let us consider the Rubio de Francia algorithm:

$$Rf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M^k\|_{L^p(u)\to L^p(u)}}.$$

Then, clearly, $f \le Rf$, $Rf \in A_1$ and using (2.1) we obtain that

$$\|Rf\|_{A_1} \le 2\|M\|_{L^p(u)\to L^p(u)} \lesssim \frac{\|u\|_{A_1}}{p-1},$$

and

$$\|Rf\|_{L^p(u)} \le 2\|f\|_{L^p(u)}.$$

Therefore,

$$\int |Tf(x)|^p u(x)dx = \int |Tf(x)|^p (Rf(x))^{\frac{p}{p_0}(p-p_0)} (Rf(x))^{\frac{p}{p_0}(p_0-p)} u(x)dx$$

$$\le \left(\int |Tf(x)|^{p_0} (Rf(x))^{p-p_0} u(x)dx\right)^{\frac{p}{p_0}} \left(\int (Rf(x))^p u(x)dx\right)^{1-\frac{p}{p_0}}$$

$$\le 2^{\frac{p(p_0-p)}{p_0}} \varphi(\|(Rf)^{p-p_0}u\|_{A_{p_0}})^p \left(\int |f(x)|^{p_0} (Rf(x))^{p-p_0} u(x)dx\right)^{\frac{p}{p_0}} \left(\int |f(x)|^p u(x)dx\right)^{1-\frac{p}{p_0}}$$

$$\lesssim \varphi(\|(Rf)^{p-p_0}u\|_{A_{p_0}})^p \left(\int |f(x)|^p u(x)dx\right),$$

and we obtain that $T : L^p(u) \to L^p(u)$ with constant less than or equal to the first term in (2.3), since, by (1.6),

$$\|(Rf)^{p-p_0}u\|_{A_{p_0}} \lesssim \|Rf\|_{A_1}^{p_0-1}\|u\|_{A_1} \lesssim \frac{\|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}}.$$

2) To get the second term in (2.3), we avoid the use of Rubio de Francia's algorithm:

$$\begin{aligned}\int |Tf(x)|^p u(x)dx &= \int |Tf(x)|^p (Mf(x))^{\frac{p}{p_0}(p-p_0)}(Mf(x))^{\frac{p}{p_0}(p_0-p)}u(x)dx \\ &\le \left(\int |Tf(x)|^{p_0}(Mf(x))^{p-p_0}u(x)dx\right)^{\frac{p}{p_0}}\left(\int (Mf(x))^p u(x)dx\right)^{1-\frac{p}{p_0}}.\end{aligned}$$

Now, by (1.4) and (1.3), we have that $w := (Mf)^{p-p_0}u \in A_{p_0}$ with

$$\begin{aligned}\|w\|_{A_{p_0}} &= \left\|\left[(Mf)^{\frac{p_0-p}{p_0-1}}\right]^{1-p_0} u\right\|_{A_{p_0}} \le \left\|(Mf)^{\frac{p_0-p}{p_0-1}}\right\|_{A_1}^{p_0-1}\|u\|_{A_1} \\ &\lesssim \left(\frac{p_0-1}{p-1}\right)^{p_0-1}\|u\|_{A_1}.\end{aligned} \tag{2.4}$$

Hence, we can use our assumption and (2.1) to deduce that

$$\begin{aligned}&\int |Tf(x)|^p u(x)dx \\ &\lesssim \varphi(\|w\|_{A_{p_0}})^p\left(\int |f(x)|^{p_0}(Mf(x))^{p-p_0}u(x)dx\right)^{\frac{p}{p_0}}\left(\left(\frac{\|u\|_{A_1}}{p-1}\right)^p\int |f(x)|^p u(x)dx\right)^{1-\frac{p}{p_0}} \\ &\le \left(\frac{\|u\|_{A_1}}{p-1}\right)^{p(1-\frac{p}{p_0})}\varphi\left(\frac{C\|u\|_{A_1}}{(p-1)^{p_0-1}}\right)^p\left(\int |f(x)|^p u(x)dx\right),\end{aligned}$$

and the result follows taking the minimum of both constants. □

We observe that, since we are interested in getting the slowest blow-up possible in terms of p, the constant obtained by the first procedure is better for our purposes. However, in standard cases ($p_0 = 2$, $\varphi(t) = t$), it gives a worst exponent in $\|u\|_{A_1}$.

As a first consequence, we obtain the following extension of Corollary 1.6 for weights $u \in A_1$.

Corollary 2.2. *Let $1 < p_0 < \infty$, $s > 0$ and let T be a sublinear operator such that*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $\|w\|_{A_{p_0}}^s$. Then, for every $u \in A_1$,

$$T : L(\log L)^{s(p_0-1)}(u) \longrightarrow E_{s(p_0-1)}(u)$$

is also bounded with constant less than or equal to $C\|u\|_{A_1}^{sp_0}$.

Remark 2.3. *We shall keep the letter s for the exponent of $\|w\|_{p_0}$ in the strong-type (p_0, p_0) case.*

2.2. **From weak-type to weak-type.** In this subsection, the starting hypothesis will be a weak type estimate at the p_0-level for every weight in the A_{p_0} class. This information will give us a weak type estimate at any level $1 < p < p_0$ for every weight $u \in A_1$ and this new information will finally lead us to the boundedness of our operator in a space near L^1.

Theorem 2.4. *Let T be an operator such that, for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, where $\varphi : [0, \infty) \to [0, \infty)$ is an increasing function. Then, for every $1 < p < p_0$ and every $u \in A_1$,

$$T : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded with constant controlled by

$$C_1\|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}}\varphi\left(C_2\left(\frac{p_0-1}{p-1}\right)^{p_0-1}\|u\|_{A_1}\right).$$

Proof. Let $\gamma > 0$ and $y > 0$,

$$\lambda^u_{Tf}(y) \le \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|Tf|>y\}}(Mf)^{p-p_0}(x)u(x)dx.$$

Then, by hypothesis, since $w = (Mf)^{p-p_0}u$ we deduce that

$$\begin{aligned}\lambda^u_{Tf}(y) &\le \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\int |f(x)|^{p_0}Mf(x)^{p-p_0}u(x)dx \\ &\le \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\int |f(x)|^{p}u(x)dx.\end{aligned}$$

Using (2.2), we have that

$$\lambda^u_{Mf}(\gamma y) \lesssim \frac{\|u\|_{A_1}}{y^p\gamma^p}\|f\|^p_{L^p(u)}.$$

Combining these two facts and multiplying by y^p we obtain

$$y^p\lambda^u_{Tf}(y) \lesssim \left(\frac{\|u\|_{A_1}}{\gamma^p} + \gamma^{p_0-p}\varphi(\|w\|_{A_{p_0}})^{p_0}\right)\|f\|^p_{L^p(u)}.$$

Finally, we can minimize the right-hand side with respect to $\gamma > 0$ by choosing $\gamma = \|u\|_{A_1}^{1/p_0}\varphi(\|w\|_{A_{p_0}})^{-1}$, and taking supremum over $y > 0$, we get,

$$\|Tf\|^p_{L^{p,\infty}(u)} \lesssim \|u\|_{A_1}^{1-p/p_0}\varphi(\|w\|_{A_{p_0}})^p\|f\|^p_{L^p(u)}.$$

This estimate, together with (2.4), completes the proof. □

Corollary 2.5. *Under the hypotheses of Theorem 2.4, with $\varphi(t) = t^\sigma$ for some $\sigma > 0$, we obtain that, for every $u \in A_1$,*

$$T : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded with constant less than or equal to

$$\frac{C\|u\|_{A_1}^{\sigma+\frac{1}{p}-\frac{1}{p_0}}}{(p-1)^{\sigma(p_0-1)}}.$$

At this point, we have to use the first variant of Yano's extrapolation theorem concerning weak-type spaces. In 1996, N. Yu Antonov [1] proved that there is almost everywhere convergence for the Fourier series of every function in $L\log L\log_3 L(\mathbb{T})$. Even though he did not write it explicitly, behind his ideas there was the following extrapolation argument (see [2, 11, 12, 39] for more details):

Theorem 2.6. *Let $1<p_0<\infty$ and $m>0$. If T is a sublinear operator such that*

$$T: L^p(\mu) \longrightarrow L^{p,\infty}(\mu) \tag{2.5}$$

is bounded with constant controlled by $(p-1)^{-m}$ for every $1<p\le p_0$, then

$$T: L(\log L)^m \log_3 L(\mu) \longrightarrow R_m(\mu)$$

is also bounded, where $R_m(\mu)$ is the space of μ-measurable functions such that

$$\|f\|_{R_m(\mu)} = \sup_{t>0} \frac{t f^*_\mu(t)}{\log_1^m(t)} < \infty.$$

Corollary 2.7. *Under the hypotheses of Theorem 2.4 with $\varphi(t)=t^\sigma$, we obtain that, for every $u\in A_1$,*

$$T: L(\log L)^{\sigma(p_0-1)} \log_3 L(u) \longrightarrow R_{\sigma(p_0-1)}(u)$$

is bounded with constant less than or equal to $C\|u\|_{A_1}^{\sigma+1-\frac{1}{p_0}}$.

Remark 2.8. *We shall keep the letter σ for the exponent of $\|w\|_{p_0}$ in the weak-type (p_0,p_0) case. Clearly, for a given operator, $\sigma\le s$.*

Remark 2.9. *It is important to mention that, in order to prove Theorem 2.6, we only need to assume that the sublinear operator T satisfies, for every function f such that $\|f\|_\infty\le 1$,*

$$\|Tf\|_{L^{p,\infty}(\mu)} \lesssim \frac{1}{(p-1)^m}\|f\|_{L^1(\mu)}^{1/p}. \tag{2.6}$$

From here, the boundedness on $L(\log L)^m\log_3 L(\mu)$ is obtained by expressing f as an appropriate linear combination of functions bounded by 1. This estimate (2.6) can be deduced from (2.5), but it also follows from the weaker boundedness

$$T: L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu),$$

with constant $(p-1)^{-m}$.

2.3. **From restricted weak-type to restricted weak-type.** For many interesting operators, the hypothesis that we have is not of weak type, but a weaker condition such as a restricted weak type estimate. That is, we only know the weak type inequality for functions of the form $f=\chi_E$. In this subsection we shall see that we can also obtain some boundedness in a space near L^1.

Theorem 2.10. *Let T be an operator such that, for some $1<p_0<\infty$ and every $w\in A_{p_0}$,*

$$T: L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$ where $\varphi:[0,\infty)\to[0,\infty)$ is an increasing function. Then, for every $1<p<p_0$ and every $u\in A_1$,

$$T: L^{p,\frac{p}{p_0}}(u) \longrightarrow L^{p,\infty}(u)$$

is bounded with constant controlled by

$$C_1\|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}}\varphi\left(C_2\left(\frac{p_0-1}{p-1}\right)^{p_0-1}\|u\|_{A_1}\right).$$

The proof follows the same pattern as in Theorem 2.4 with the obvious modifications. However, since the result is not the one we can initially expect and we have to be careful with the behavior of the constant, we include the details.

Proof. Let $\gamma>0$ and $y>0$. Then, if $w=(Mf)^{p-p_0}u$,

$$\begin{aligned}\lambda^u_{Tf}(y) &\leq \lambda^u_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|Tf|>y\}}w(x)dx\\ &\lesssim \lambda^u_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\left(\int_0^\infty\left(\int_{\{|f|>z\}}w(x)dx\right)^{1/p_0}dz\right)^{p_0}.\end{aligned}$$

But, since $p-p_0<0$, we can bound $w=(Mf)^{p-p_0}u\leq z^{p-p_0}u$ on the set $\{|f|>z\}$, so we conclude that

$$\begin{aligned}\lambda^u_{Tf}(y) &\lesssim \lambda^u_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\left(\int_0^\infty z^{\frac{p}{p_0}-1}\left(\int_{\{|f|>z\}}u(x)dx\right)^{1/p_0}dz\right)^{p_0}\\ &\approx \lambda^u_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\|f\|^p_{L^{p,\frac{p}{p_0}}(u)}.\end{aligned}$$

The result follows as in Theorem 2.4, using (2.2) and the fact that $\|f\|_{L^p(u)}\leq\|f\|_{L^{p,\frac{p}{p_0}}(u)}$. □

Lemma 2.11. *If T is a sublinear operator such that, for every $1<p<p_0$,*

$$T: L^{p,\frac{p}{p_0}}(\mu)\longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant less than or equal to $(p-1)^{-m}$, then:

(i) *It holds that*

$$T: L^{p,1}(\mu)\longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant less than or equal to $C(p-1)^{-m-1}$.

(ii) *For every function f such that $\|f\|_\infty\leq 1$,*

$$\|Tf\|_{L^{p,\infty}(\mu)}\lesssim\frac{1}{(p-1)^{m+\frac{p_0-1}{p}}}\|f\|^{1/p}_{L^1(\mu)}.$$

Proof. (i) The first result is well-known since, for every measurable set E,

$$\|T\chi_E\|_{L^{p,\infty}(\mu)}\lesssim\frac{1}{(p-1)^m}\mu(E), \tag{2.7}$$

and $L^{p,\infty}(\mu)$ can be endowed with a norm $\|\cdot\|_*$ such that

$$\|f\|_{L^{p,\infty}(\mu)}\leq\|f\|_*\leq\frac{1}{p-1}\|f\|_{L^{p,\infty}(\mu)},$$

and hence the above estimate over measurable sets can be extended to any function by adding a factor $\frac{1}{p-1}$ to the constant.
(ii) Let f be bounded by 1. Then

$$\begin{aligned}\|f\|_{L^{p,\frac{p}{p_0}}(\mu)} &= \left(p\int_0^1 \lambda_f^\mu(t)^{1/p_0} t^{\frac{p}{p_0}-1}dt\right)^{p_0/p} \lesssim \left(\int_0^1 \lambda_f^\mu(t)dt\right)^{1/p}\left(\int_0^1 t^{\frac{p-p_0}{p_0-1}}dt\right)^{\frac{p_0-1}{p}}\\ &\lesssim \|f\|_{L^1(\mu)}^{1/p}\left(\frac{p_0-1}{p-1}\right)^{\frac{p_0-1}{p}},\end{aligned}$$

and the result follows. □

As a consequence of Theorem 2.10, Lemma 2.11 and Remark 2.9, we obtain the following endpoint estimate:

Corollary 2.12. *Let $1 < p_0 < \infty$, $r > 0$ and let T be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $\|w\|_{A_{p_0}}^r$. Then, for every $u \in A_1$,

$$T : L(\log L)^\beta \log_3 L(u) \longrightarrow R_\beta(u)$$

is also bounded with constant $C\|u\|_{A_1}^{r+1-\frac{1}{p_0}}$, where $\beta = r(p_0-1) + \min(1, p_0-1)$.

Remark 2.13. *We shall keep the letter r for the exponent of $\|w\|_{p_0}$ in the restricted weak-type (p_0, p_0) case. Clearly, for a given operator, $r \le \sigma \le s$.*

Remark 2.14. *There is large class of operators (called (ε, δ)-atomic approximable, see Definition 3.2 and* [9, 10]*) for which an estimate of the form* (2.7) *implies that, for every function f bounded by* 1*,*

$$\|Tf\|_{L^{p,\infty}(u)} \lesssim \frac{1}{(p-1)^m}\|f\|_{L^1(u)}^{1/p}.$$

Hence, by Remark 2.9, we obtain that if an operator T in this class satisfies the hypotheses of Corollary 2.12, then

$$T : L(\log L)^{r(p_0-1)} \log_3 L(u) \longrightarrow R_{r(p_0-1)}(u).$$

2.4. **From strong weak-type to strong weak-type.** The motivation of this subsection is the following: in [13], it was proved that one can improve the endpoint space obtained by the classical Yano's extrapolation or even by Antonov's extrapolation by assuming a stronger condition on T; namely that the operator is of strong type on the bigger space $L^{p,\infty}$. Hence, we want to apply our technique to the case of operators when the starting hypothesis is precisely to have a strong type estimate at level $L^{p_0,\infty}$ with respect to every A_{p_0} weight with the hope of improving the final boundedness near L^1. We shall see that this is the case.

Theorem 2.15. *Let T be an operator such that, for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, where $\varphi : [0,\infty) \to [0,\infty)$ is an increasing function. Then, for every $1 < p < p_0$ and every $u \in A_1$,

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u)$$

is bounded with constant

$$C_1\varphi\left(\frac{C_2\|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}}\right).$$

Proof. We use Rubio de Francia's algorithm as follows:

$$Rf(x)=\sum_{k=0}^{\infty}\frac{M^k f(x)}{2^k\|M^k\|_{L^{p,\infty}(u)\to L^{p,\infty}(u)}}.$$

Then, clearly, $f\le Rf$, $Rf\in A_1$ and using (2.1) we obtain that

$$\|Rf\|_{A_1}\le 2\|M\|_{L^{p,\infty}(u)\to L^{p,\infty}(u)}\lesssim\frac{\|u\|_{A_1}}{p-1}$$

and

$$\|Rf\|_{L^{p,\infty}(u)}\le 2\|f\|_{L^{p,\infty}(u)}.$$

Let $\gamma>0$ and $y>0$,

$$\lambda^u_{Tf}(y)\le\lambda^u_{Rf}(\gamma y)+\gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|Tf|>y\}}(Rf)^{p-p_0}(x)u(x)dx.$$

Then, by hypothesis, we deduce that if $w=(Rf)^{p-p_0}u$,

$$\begin{aligned}\lambda^u_{Tf}(y)&\le\lambda^u_{Rf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\sup_{z>0}z^{p_0}\int_{\{|f|>z\}}Rf(x)^{p-p_0}u(x)dx\\&\le\lambda^u_{Rf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p}\sup_{z>0}z^{p}\int_{\{|f|>z\}}u(x)dx.\end{aligned}$$

Using that

$$\lambda^u_{Rf}(\gamma y)\lesssim\frac{2^p}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}(u)},$$

we obtain

$$y^p\lambda^u_{Tf}(y)\lesssim\left(\frac{1}{\gamma^p}+\gamma^{p_0-p}\varphi(\|w\|_{A_{p_0}})^{p_0}\right)\|f\|^p_{L^{p,\infty}(u)},$$

and the result follows minimizing with $\gamma=\varphi(\|w\|_{A_{p_0}})^{-1}$ and using the estimate analogous to (2.4) with R instead of M. □

The third variant of Yano's extrapolation theorem we need is the following:

Theorem 2.16 ([13])**.** *Let μ be a σ-finite, non-atomic measure, $1<p_0<\infty$, $m>0$, and let*

$$T:L^{p,\infty}(\mu)\longrightarrow L^{p,\infty}(\mu)$$

be a bounded sublinear operator with constant controlled by $(p-1)^{-m}$ for every $1<p\le p_0$. Then,

$$T:\left[L(\log L)^{m-1}\log_3 L(\mu)\right]_1\longrightarrow R_m(\mu)$$

is bounded, where $X=\left[L(\log L)^{m-1}\log_3 L(\mu)\right]_1$ is the set of measurable functions such that

$$\|f\|_X=\|f\|_{L^{1,\infty}(\mu)}+\int_0^1\frac{\sup_{t\le y}tf^*_\mu(t)}{y}\Big(\log_1\Big(\frac{1}{y}\Big)\Big)^{m-1}\log_3\Big(\frac{1}{y}\Big)dy<\infty.$$

Remark 2.17. *Here, we have to emphasize that*

$$L(\log L)^m \log_3 L(\mu) \subsetneq \left[L(\log L)^{m-1} \log_3 L(\mu)\right]_1.$$

Thus, except for the $\log_3 L$ *factor, we can say that in order to obtain the best estimate from a Yano's extrapolation type theorem, it would be convenient to compute the best strong weak-type* (p,p) *constant for* T *as* $p \to 1$.

Corollary 2.18. *Let* $1 < p_0 < \infty$, *and let* T *be a sublinear operator such that*

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded for every $w \in A_{p_0}$ *with constant* $\|w\|_{A_{p_0}}^{\alpha}$. *Then, for every* $u \in A_1$,

$$T : \left[L(\log L)^{\alpha(p_0-1)-1} \log_3 L(u)\right]_1 \longrightarrow R_{\alpha(p_0-1)-1}(u)$$

is also bounded with constant less than or equal to $C\|u\|_{A_1}^{\alpha p_0}$.

Remark 2.19. *We shall keep the letter* α *for the exponent of* $\|w\|_{p_0}$ *in the strong weak-type* (p_0,p_0) *case. Clearly, for a given operator,* $r \le \sigma \le \alpha$, *but there is no clear relation between* α *and* s.

2.5. **From restricted weak-type to strong weak-type.** Taking into account Remarks 2.13 and 2.17, it is clear that a good way to obtain the best endpoint estimate would be to seek good restricted weak-type estimates for T with respect to every weight in A_{p_0} (that is, a small exponent r) and from here, try to find a good constant for the strong weak-type (p,p) with respect to A_1 weights. In other words, we want to find an optimal way to relate r and δ in the following situation:

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w), \qquad C\|w\|_{A_{p_0}}^{r} \implies T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u), \qquad \frac{C}{(p-1)^{\delta}}.$$

To this end, we shall use the following interpolation result which is proved in the Appendix.

Lemma 2.20. *Let* $0 < s_0, s_1 \le 1 < q_0 < q_1 < \infty$ *and let* T *be a sublinear operator such that, for some weight* u,

$$T : L^{q_j,s_j}(u) \longrightarrow L^{q_j,\infty}(u)$$

is bounded with constant M_j, *for* $j = 0,1$. *Then, for every* $0 < \theta < 1$, *if* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, *we have that*

$$T : L^{q,\infty}(u) \longrightarrow L^{q,\infty}(u)$$

is bounded with constant controlled by $BM_0^{1-\theta}M_1^{\theta}$, *where*

$$B = \left(\frac{q_0 q}{s_0(q-q_0)}\right)^{1/s_0} + \left(\frac{q_1 q}{s_1(q_1-q)}\right)^{1/s_1} + \left(\frac{q_1}{s_1}\right)^{1/s_1}$$

Theorem 2.21. *Let* $1 < p_0 < \infty$, *and let* T *be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded for every $w \in A_{p_0}$ *with constant* $\varphi(\|w\|_{A_{p_0}})$, *where* $\varphi : [0,\infty) \to [0,\infty)$ *is an increasing function. Then, for every* $w \in A_{p_0}$

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded with constant less than or equal to $C_1\|w\|_{A_{p_0}}^{p_0'-1}\varphi(C_2\|w\|_{A_{p_0}})$.

Proof. From our hypothesis, we can write that, for every measurable set $E \subseteq \mathbb{R}^n$ and every $w \in A_{p_0}$,

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \le \varphi(\|w\|_{A_{p_0}})w(E)^{1/p_0}.$$

This estimate can be extrapolated up by means of the modern version of Rubio de Francia's theorem (see its statement in [22]) and, given $\varepsilon > 0$, we obtain that

$$\|T\chi_E\|_{L^{p_0+\varepsilon,\infty}(w)} \lesssim \varphi(C\|w\|_{A_{p_0+\varepsilon}})w(E)^{1/(p_0+\varepsilon)},$$

for $w \in A_{p_0+\varepsilon}$. In particular,

$$T : L^{p_0+\varepsilon,1}(w) \longrightarrow L^{p_0+\varepsilon,\infty}(w), \tag{2.8}$$

for every $w \in A_{p_0}$ and constant controlled by

$$\frac{\varphi(C\|w\|_{A_{p_0}})}{p_0+\varepsilon-1} \lesssim \varphi(C\|w\|_{A_{p_0}}).$$

Now we want to extrapolate down to $p_0 - \varepsilon$. Fix $w \in A_{p_0}$ and set $\varepsilon = C(p_0-1)\|w\|_{A_{p_0}}^{1-p_0'}$ in such a way that $w \in A_{p_0-2\varepsilon}$ and $\|w\|_{A_{p_0-2\varepsilon}} \lesssim \|w\|_{A_{p_0}}$ (see [29]). Now we proceed as in Theorem 2.10. For every measurable set $E \subseteq \mathbb{R}^n$, $\gamma > 0$ and $y > 0$,

$$\lambda^w_{T\chi_E}(y) \le \lambda^w_{M\chi_E}(\gamma y) + \gamma^\varepsilon y^\varepsilon \int_{\{|T\chi_E|>y\}} (M\chi_E)^{-\varepsilon}(x)w(x)dx.$$

But, using [22, Lemma 2.1],

$$\|(M\chi_E)^{-\varepsilon}w\|_{A_{p_0}} \le \|(M\chi_E)^{1/2}\|_{A_1}^{2\varepsilon}\|w\|_{A_{p_0-2\varepsilon}} \lesssim \|w\|_{A_{p_0}},$$

so we can use our hypothesis and (2.2) to deduce that

$$y^{p_0-\varepsilon}\lambda^w_{T\chi_E}(y) \lesssim \left(\frac{\|w\|_{A_{p_0}}}{\gamma^{p_0-\varepsilon}} + \gamma^\varepsilon\varphi(\|w\|_{A_{p_0}})^{p_0}\right)w(E).$$

Minimizing in $\gamma > 0$ we conclude that

$$\|T\chi_E\|_{L^{p_0-\varepsilon,\infty}(w)} \lesssim \|w\|_{A_{p_0}}^{\frac{\varepsilon}{p_0(p_0-\varepsilon)}}\varphi(\|w\|_{A_{p_0}})w(E)^{1/(p_0-\epsilon)},$$

and hence

$$T : L^{p_0-\varepsilon,1}(w) \longrightarrow L^{p_0-\varepsilon,\infty}(w) \tag{2.9}$$

with constant

$$\frac{\|w\|_{A_{p_0}}^{\frac{\varepsilon}{p_0(p_0-\varepsilon)}}\varphi(\|w\|_{A_{p_0}})}{p_0-\varepsilon-1} \lesssim \varphi(\|w\|_{A_{p_0}}),$$

recalling that $\varepsilon \approx (p_0-1)\|w\|_{A_{p_0}}^{1-p_0'}$. Finally, we use Lemma 2.20 with (2.9) and (2.8) to conclude that

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

with constant controlled by $\frac{C\varphi(C\|w_{A_{p_0}})}{\varepsilon} \approx C_{p_0}\|w\|_{A_{p_0}}^{p_0'-1}\varphi(C\|w\|_{A_{p_0}})$. □

Corollary 2.22. *Let $1 < p_0 < \infty$, and let T be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $\|w\|_{A_{p_0}}^r$. Then, for every $u \in A_1$,

$$T : \Big[L(\log L)^{r(p_0-1)} \log_3 L(u)\Big]_1 \longrightarrow R_{r(p_0-1)}(u)$$

is also bounded with constant less than or equal to $C\|u\|_{A_1}^{p_0'+rp_0}$.

3. A different class of weights

Very recently, the authors in [10] presented a different kind of extrapolation in the sense of Rubio de Francia that allows us to reach the endpoint $p = 1$ at least when restricted to characteristic functions. More precisely, they introduced a new class of weights $\widehat{A}_p$, closely related to A_p, in such a way that, if an operator T satisfies

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w), \tag{3.1}$$

for some $1 < p_0 < \infty$ and every weight $w \in \widehat{A}_{p_0}$, then we can conclude that

$$\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E),$$

for every $u \in A_1$ and every measurable set $E \subseteq \mathbb{R}^n$. The class $\widehat{A}_p$ is obviously larger than A_p, because otherwise we would have, again, M^2 as a counterexample. However, it holds that, for every $1 \leq p < \infty$ and every $\varepsilon > 0$,

$$A_p \subseteq \widehat{A}_p \subseteq A_{p+\varepsilon}.$$

The precise definition of these classes is

$$\widehat{A}_p = \{(Mf)^{1-p}u : f \in L^1_{\text{loc}},\, u \in A_1\},$$

and they are a subclass of the so-called $A_p^{\mathcal{R}}$ weights, introduced in 1982 by R. Kerman and A. Torchinsky [30] to characterize the boundedness (3.1) of the Hardy-Littlewood maximal operator M. The extrapolation presented in [10] is more general, but the most interesting part for our purposes can be stated as follows:

Theorem 3.1 ([10])**.** *Let T be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w).$$

Then, for every $u \in A_1$,

(i) $\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E), \quad E \subseteq \mathbb{R}^n$,
(ii) $T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\text{loc}}(u), \quad \varepsilon > 0$.

It is known that, in general, the estimate (i) cannot hold for every function $f \in L^1(u)$. Take, for instance, the operator

$$Af(x) = \left\| \frac{f(x+y)}{y} \right\|_{L^{1,\infty}(\mathbb{R})}, \tag{3.2}$$

which was introduced in [3] and is related to Bourgain's return time theorems. It trivially satisfies $A\chi_E \le M\chi_E$ (and hence, the assumption of Theorem 3.1), but it is not of weak-type $(1,1)$. However, it can be proved that, for a wide class of operators called (ε,δ)-atomic approximable (see Definition 3.2 below), the estimate on characteristic functions (i) is in fact equivalent to the weighted weak-type $(1,1)$.

Definition 3.2. *Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-atom if it satisfies the following properties:*

- $\int_{\mathbb{R}^n} a = 0$, *and*
- *there exists a cube Q such that $|Q| \le \delta$ and $\operatorname{supp} a \subseteq Q$.*

With this, a sublinear operator T is (ε,δ)-atomic if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|Ta\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \le \varepsilon \|a\|_{L^1(\mathbb{R}^n)},$$

for every δ-atom a, and T is said to be (ε,δ)-atomic approximable if there exists a sequence $\{T_n\}_n$ of (ε,δ)-atomic operators such that, for every measurable set E, $|T_n\chi_E| \le |T\chi_E|$ and, for every function $f \in L^1(\mathbb{R}^n)$ with $\|f\|_\infty \le 1$,

$$|Tf(x)| \le \liminf_n |T_n f(x)|, \quad \text{a.e. } x \in \mathbb{R}^n.$$

In [9], the author shows that this is not a strong property to assume on an operator. For instance, it is checked that if

$$Tf(x) = K * f(x), \tag{3.3}$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \le p < \infty$, then T is (ε,δ)-atomic, and if $\{T_n\}_n$ is a sequence of (ε,δ)-atomic operators, then $\sup_n |T_n f(x)|$ is (ε,δ)-atomic approximable. See [9, 10] for more examples.

Therefore, the conclusion (ii) of Theorem 3.1 is especially interesting for operators which are not (ε,δ)-atomic approximable, since in this case, (ii) is the best endpoint result that is not restricted to characteristic functions. In this section, we will see that this can be improved to the larger space $L\log_2 L(u)$.

Theorem 3.3. *Let T be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded. Then, for every $u \in A_1$,

$$T : L\log_2 L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$$

is also bounded.

Proof. First, we apply Theorem 3.1, which yields

(i) $\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E)$,
(ii) $T : L(\log L)^\varepsilon(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$.

Since $L^1(u) \cap L^\infty$ is continuously embedded in $L(\log L)^\varepsilon(u)$, from (ii) we deduce that,

$$T : L^1(u) \cap L^\infty \to L^{1,\infty}_{\mathrm{loc}}(u). \tag{3.4}$$

Take now a non-negative function $f = f_0 + f_1$, with $f_0 = f\chi_{\{f\leq 1\}}$ and $f_1 = f\chi_{\{f>1\}}$. By sublinearity, we have that $\|Tf\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|Tf_0\|_{L^{1,\infty}_{\text{loc}}(u)} + \|Tf_1\|_{L^{1,\infty}_{\text{loc}}(u)}$. For the term with f_0, we use (3.4) and $\|f_0\|_\infty \leq 1$ to get

$$\|Tf_0\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|f_0\|_{L^1(u)} + \|f_0\|_\infty \leq \|f\|_{L\log_2 L(u)} + 1.$$

Now, to deal with f_1, we need to resort to [40, Lemma 4], where the author presents the following decomposition for non-negative functions:

$$f_1(x) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} 2^k \chi_{E_{k,j}}(x) \quad \text{a.e. } x\in\mathbb{R}^n, \tag{3.5}$$

where the sets $E_{k,j}$ depend on f_1 and are defined in such a way that, for every weight (in particular u),

$$u(E_{k,j}) \leq \lambda^u_{f_1}(2^{k+j}).$$

For every $N > 0$, set f_1^N to be the following truncated series:

$$f_1^N(x) = \sum_{j=1}^{N}\sum_{|k|\leq N} 2^k \chi_{E_{k,j}}(x).$$

Next, since it is a finite sum, we use the sublinearity of T and (i) in order to obtain

$$\|Tf_1^N\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \sum_{j=1}^{\infty} \log_1(j) \sum_{k\in\mathbb{Z}} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}). \tag{3.6}$$

The logarithmic terms come from Stein and Weiss' lemma for the $L^{1,\infty}$ quasinorm (see [41]). Now, fix $j \geq 1$ and split the inner sum into three pieces: $I_j^1 + I_j^2 + I_j^3$. The first one will be

$$I_j^1 = \sum_{k<-j} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}) \leq \|f\|_{L\log_2 L(u)} \sum_{k=j+1}^{\infty} 2^{-k}\log_1(k).$$

Here we used that, since $f_1 > 1$, we have $\lambda^u_{f_1}(2^{k+j}) \leq \|f\|_{L\log_2 L(u)}$ whenever $k < -j$, because in this case,

$$\lambda^u_{f_1}(2^{k+j}) = \lambda_{f_1}(1) \leq \|f\|_{L\log_2 L(u)}.$$

The second term we need to consider is

$$\begin{aligned} I_j^2 &= \sum_{k=-j}^{0} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}) = 2^{-j}\sum_{k=-j}^{0} 2^{k+j}\log_1(|k|)\lambda^u_{f_1}(2^{k+j}) \\ &\leq 2^{-j}\|f\|_{L\log_2 L(u)} \sum_{k=0}^{j}\log_1(k). \end{aligned}$$

Here we just used that $t\lambda_f^u(t) \leq \|f\|_{L\log_2 L(u)}$, for every $t > 0$. Finally,

$$I_j^3 = \sum_{k=1}^{\infty} 2^k \log_1(k)\lambda_{f_1}^u(2^{k+j}) \leq 2^{-j} \sum_{k=1}^{\infty} 2^{k+j} \log_1(k+j)\lambda_{f_1}^u(2^{k+j})$$
$$\lesssim 2^{-j} \int_0^{\infty} \lambda_f^u(s) \log_2(s) ds \lesssim 2^{-j} \|f\|_{L\log_2 L(u)}.$$

Now we go back to (3.6) and using the bounds for I_j^m, $m = 1, 2, 3$, we conclude that

$$\|Tf_1^N\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} \sum_{j=1}^{\infty} \log_1(j) \left(\sum_{k=j+1}^{\infty} 2^{-k} \log_1(k) + 2^{-j} \sum_{k=0}^{j} \log_1(k) + 2^{-j} \right)$$
$$\lesssim \|f\|_{L\log_2 L(u)}.$$

Therefore, we have that $\|Tf_1^N\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)}$. If we show that f_1^N converges to f_1 in $L\log_2 L(u)$, then we conclude $\|Tf_1\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)}$ and hence,

$$\|Tf\|_{L^{1,\infty}_{\text{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} + 1.$$

From here, we finish the proof changing f by αf and letting α tend to infinity. To show that $f_1^N \to f_1$ in $L\log_2 L(u)$, we observe that the difference $f_1(x) - f_1^N(x)$ decreases to zero for almost every $x \in \mathbb{R}^n$, since f_1^N is a partial sum of a convergent series of positive terms that coincides with f_1 almost everywhere. In particular, its decreasing rearrangement with respect to u satisfies that

$$(f_1 - f_1^N)_u^*(t) \longrightarrow 0, \quad \text{a.e. } t \in (0, \infty).$$

On the other hand, $|f_1 - f_1^N|$ can be pointwise controlled by $f_1 \in L\log_2 L(u)$, so

$$\left| (f_1 - f_1^N)_u^*(t) \log_2 \frac{1}{t} \right| \leq (f_1)_u^*(t) \log_2 \frac{1}{t} \in L^1(0, \infty).$$

Therefore, by the dominated convergence theorem,

$$\|f_1 - f_1^N\|_{L\log_2 L(u)} = \int_0^{\infty} (f_1 - f_1^N)_u^*(t) \log_2 \frac{1}{t} dt \longrightarrow 0,$$

as $N \to \infty$, so we finish the proof. □

4. Examples and Applications

4.1. **Composition of Rubio de Francia operators.** Many times, we are interested in operators T that can be expressed as a composition of other (simpler) ones, say $T \approx T_1 \circ T_2$. It is clear that if we have boundedness information for T_1 and T_2, and it can be put together as a composition, then we can draw conclusions for the original operator T. We will exemplify this with Calderón-Zygmund operators, although the same argument can be carried out with other operators for which their strong and weak-type (p, p) boundedness constants are known. We will also present an application to the composition of commutators of general linear operators.

A Calderón-Zygmund operator is an $L^2(\mathbb{R}^n)$ bounded integral operator T whose kernel satisfies certain standard growth and smoothness conditions. This definition includes the

Hilbert, Beurling and Riesz transforms, among others, and it is well-known they are of strong-type (p,p) for every $1<p<\infty$ and every A_p weight. However, the sharp dependence of the boundedness constant on the weight was the result of several years of research culminating in [27]. In [28], the authors extended the result to maximal Calderón-Zygmund operators and included the weak-type case. If K is the kernel of a Calderón-Zygmund operator T, then we define

$$T_*f(x)=\sup_{0<\varepsilon<\delta}\left|\int_{\varepsilon<|y|<\delta}K(x,y)f(y)dy\right|,$$

and the main result in [28] states that

- $\|T_*\|_{L^p(w)\to L^{p,\infty}(w)}\lesssim\|w\|_{A_p},\quad 1<p<2,$
- $\|T_*\|_{L^p(w)\to L^p(w)}\lesssim\|w\|_{A_p}^{\frac{1}{p-1}},\quad 1<p<2.$

Calderón-Zygmund operators and their maximal versions are known to be of weak-type (1,1) for A_1 weights. However, if we consider a family $\{T_i\}_{i=1}^k$, with $k\geq 2$, this need not be true for the composition $T_1\circ\cdots\circ T_k$. By iteration of the previous estimates, we have that

- $\|T_{1,*}\circ\cdots\circ T_{k,*}\|_{L^p(w)\to L^{p,\infty}(w)}\lesssim\|w\|_{A_p}^{1+\frac{k-1}{p-1}},\quad 1<p<2,$
- $\|T_{1,*}\circ\cdots\circ T_{k,*}\|_{L^p(w)\to L^p(w)}\lesssim\|w\|_{A_p}^{\frac{k}{p-1}},\quad 1<p<2.$

Using Corollary 2.7 on the weak-type estimate, we conclude the following:

Corollary 4.1. *Given a family of Calderón-Zygmund operators $\{T_i\}_{i=1}^k$, with $k\geq 2$, and $u\in A_1$, it holds that, for every $\varepsilon>0$,*

$$T_{1,*}\circ\cdots\circ T_{k,*}:L(\log L)^{k-1+\varepsilon}\log_3 L(u)\longrightarrow R_{k-1+\varepsilon}(u).$$

Notice that if we had used the strong-type estimate with Corollary 2.2, we would have gotten boundedness on $L(\log L)^k(u)$, which is a smaller space.

Our next example will seek endpoint estimates for the composition of commutators. First, recall that a locally integrable function $b:\mathbb{R}^n\to\mathbb{R}$ is said to be in BMO if

$$\|b\|_{BMO}=\sup_Q\frac{1}{|Q|}\int_Q|b(y)-b_Q|dy<\infty,$$

where $b_Q=\frac{1}{|Q|}\int_Q b$ is the average of b on the cube Q. Given an operator T and a BMO function b, we define the commutator

$$[b,T]f=bTf-T(bf).$$

We can also define the k-th order commutator as $T_b^k=[b,T_b^{k-1}]$ for every $k\geq 1$, being $T_b^0=T$. The special case when T is a Calderón-Zygmund operator was first considered in [15], and in [36] it was shown that $[b,T]$ is not of weak-type (1,1). This motivates the study of endpoint estimates for commutators. In our case, however, we will deal with general linear operators T and their commutators with BMO functions, in the spirit of [14]. More precisely, we will make use of the following result, that can be found in [14, Corollary 3.2] for $p_0=2$ and extended to $1<p_0<\infty$ by obvious modifications. Also, the case $k=1$ can be found in [35].

Theorem 4.2. *Let T be a linear operator, $1 < p_0 < \infty$ and $b \in BMO$. If*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $C\|w\|_{A_{p_0}}^s$, then, for every $k \geq 1$ and every $w \in A_{p_0}$,

$$T_b^k : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with constant less than or equal to $C_k \|b\|_{BMO}^k \|w\|_{A_{p_0}}^{k \max\left\{1, \frac{1}{p_0-1}\right\}+s}$.

In view of this behavior, a direct application of Corollary 2.2 produces the following weighted endpoint result for the k-th order commutator and A_1 weights.

Corollary 4.3. *Let T be a linear operator, $1 < p_0 < \infty$ and $b \in BMO$. If*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every $w \in A_{p_0}$ with constant $C\|w\|_{A_{p_0}}^s$, then, for every $k \geq 1$ and every $u \in A_1$,

$$T_b^k : L(\log L)^{k \max\{p_0-1,1\}+s(p_0-1)}(u) \longrightarrow E_{k \max\{p_0-1,1\}+s(p_0-1)}(u)$$

is also bounded with constant less than or equal to $C_k \|b\|_{BMO}^k \|u\|_{A_1}^{k \max\{p_0, p_0'\}+sp_0}$.

4.2. **The Carleson maximal operator.** Our next application will yield an A_1 weighted endpoint estimate for the Carleson operator, defined by

$$\mathcal{C}f(x) = \sup_{a \in \mathbb{R}} \left| \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{e^{2\pi i a y} f(y)}{x-y} dy \right|.$$

Our argument will be based on a restricted weak-type (p,p) estimate for $\mathcal{C}$ with respect to A_1 weights.

Proposition 4.4. *Given $1 < p \leq 2$ and $u \in A_1$, it holds that, for every measurable set $E \subseteq \mathbb{R}$,*

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)} \leq \frac{C_u}{p-1} u(E)^{1/p},$$

for some $C_u > 1$ depending on the weight.

Proof. In [26], the authors prove the following good-λ inequality for $\mathcal{C}$:

$$|\{x \in I_j : \mathcal{C}f(x) > 3\lambda, M_p f(x) \leq \gamma\lambda\}| \leq C \|\mathcal{C}\|_{L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})}^p \gamma^p |I_j|,$$

for every $\gamma, \lambda > 0$, $p > 1$, and with $\{I_j\}_j$ being a family of disjoint open intervals such that

$$\{\mathcal{C}f > \lambda\} = \bigcup_j I_j.$$

As usual, the operator M_p is defined by $M_p f = (M(f^p))^{1/p}$. If we only consider characteristic functions $f = \chi_E$, we can go over the proof in [26] and check that we can replace $\|\mathcal{C}\|_{L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})}$ by $\frac{C}{p-1}$, using the well-known estimate by R. Hunt [25]:

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(\mathbb{R})} \lesssim \frac{|E|^{1/p}}{p-1}, \quad 1 < p \leq 2. \tag{4.1}$$

Therefore, we get that, for every measurable set $E \subseteq \mathbb{R}$,

$$|\{x \in I_j : \mathcal{C}\chi_E(x) > 3\lambda, M\chi_E(x) \leq \gamma^p \lambda^p\}| \leq \frac{C\gamma^p}{(p-1)^p}|I_j|.$$

Now, define

$$B = \{\mathcal{C}\chi_E > 3\lambda, M\chi_E \leq \gamma^p\lambda^p\},$$

and pick $\gamma > 0$ in such a way that $\frac{C\gamma^p}{(p-1)^p} = \varepsilon$, for some $\varepsilon > 0$ to be chosen later. Then, we can just write

$$|I_j \cap B| \leq \varepsilon |I_j|.$$

Using the sharp Reverse Hölder property of A_1 weights (see [37]), we know that, for $\delta = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$,

$$u(I_j \cap B) \leq u^\delta(I_j)^{1/\delta}|I_j \cap B|^{1/\delta'} \leq 2u(I_j)|I_j|^{-1/\delta'}|I_j \cap B|^{1/\delta'} \leq 2\varepsilon^{1/\delta'}u(I_j).$$

With this estimate and recalling the definition of the intervals I_j, we have that

$$\begin{aligned} u(\{\mathcal{C}\chi_E > 3\lambda\}) &\leq \sum_j u(I_j \cap B) + u(\{M\chi_E > \gamma^p\lambda^p\}) \\ &\leq 2\varepsilon^{1/\delta'}u(\{\mathcal{C}\chi_E > \lambda\}) + u(\{M\chi_E > \gamma^p\lambda^p\}), \end{aligned}$$

and hence,

$$\|\mathcal{C}\chi_E\|^p_{L^{p,\infty}(u)} \leq 3^p 2\varepsilon^{1/\delta'}\|\mathcal{C}\chi_E\|^p_{L^{p,\infty}(u)} + 3^p\gamma^{-p}\|M\chi_E\|_{L^{1,\infty}(u)}.$$

Now we choose $\varepsilon > 0$ such that $3^p 2\varepsilon^{1/\delta'} = 1/2$, and using (2.2),

$$\|\mathcal{C}\chi_E\|^p_{L^{p,\infty}(u)} \leq 2 \cdot 3^p\gamma^{-p}\|u\|_{A_1}u(E).$$

To conclude, we only need to recall the value of γ, ε and δ to write

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)} \leq \frac{C^{1/p}2^{1/p} \cdot 3(4 \cdot 3^p)^{\frac{1+2^{n+1}\|u\|_{A_1}}{p}}\|u\|^{1/p}_{A_1}}{p-1}u(E)^{1/p} = \frac{C_u}{p-1}u(E)^{1/p}.$$

□

Theorem 4.5. *For every $u \in A_1$, we have that*

$$\mathcal{C} : L\log L\log_3 L(u) \longrightarrow R_1(u)$$

is bounded.

Proof. First, notice that the Carleson operator $\mathcal{C}$ is (ε, δ)-atomic approximable. The easiest way to check this is to recall that

$$\mathcal{C}f(x) \approx f(x) + \sup_{R\in\mathbb{Q}}\left|\int_{-R}^{R}\widehat{f}(\xi)e^{2\pi i x\xi}d\xi\right| = f(x) + \sup_{R\in\mathbb{Q}}|S_R f(x)|,$$

where S_R is of convolution type with kernel in $L^2(\mathbb{R})$, for every $R \in \mathbb{Q}$. Hence, by (3.3), $\{S_R\}_{R\in\mathbb{Q}}$ is a sequence of (ε, δ)-atomic operators and we prove our claim. The result now follows by Remarks 2.9 and 2.14 together with the estimate in Proposition 4.4. □

Remark 4.6. *Notice that, in Proposition 4.4, we could have worked with the good-λ inequality for general functions (and hence, with the norm $\|\mathcal{C}\|_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})}$). In this case, we would have gotten weak-type (p,p) estimates instead of restricted weak-type ones. However, the fact that $\mathcal{C}$ is (ε,δ)-atomic approximable and that it satisfies* (4.1) *with a good constant, makes that, in terms of extrapolation, it is more interesting to deal with characteristic functions. Theorem 4.5 extends Antonov's enpoint estimate* [1] *for $\mathcal{C}$ and shows that the Fourier integral on $\mathbb{R}$ is pointwise convergent for every function $f\in L\log L\log_3 L(u)$ with $u\in A_1$.*

4.3. **Maximally modulated singular integrals.** In view of the previous subsection, we can prove the following general result that only relies on a certain good-λ inequality. As we shall see, a wide class of operators called maximally modulated singular integrals will fall within the scope of this result (see [24, 19]).

Theorem 4.7. *Assume that T is a sublinear operator such that, for an increasing function ψ on $[1,\infty)$, it satisfies*

$$|\{x\in Q_j : Tf(x)>3\lambda, M_pf(x)\le\gamma\lambda\}|\le\psi\left(\frac{1}{p-1}\right)^p\gamma^p|Q_j|,$$

for every $\gamma,\lambda>0$, $1<p\le p_0$, and with $\{Q_j\}_j$ being a family of disjoint open cubes covering $\{Tf>\lambda\}$. Alternatively, we can assume that the same holds only on characteristic functions $f=\chi_E$ with a function ψ_r instead of ψ. Then, when $u\in A_1$,

- *$T: L^p(u)\to L^{p,\infty}(u)$ is bounded with constant controlled by $C_u\psi\left(\frac{1}{p-1}\right)$.*
- *It holds that*

$$\|T\chi_E\|_{L^{p,\infty}(u)}\le C_u\psi_r\left(\frac{1}{p-1}\right)u(E)^{1/p}.$$

Naturally, depending on the expression of ψ (or ψ_r), and whether T is (ε,δ)-atomic or not, one can try to extrapolate these estimates for particular examples of T, as we did in Theorem 4.5 for the Carleson operator $\mathcal{C}$ with $\psi_r=\mathrm{Id}$. As we anticipated, other examples to which we can apply Theorem 4.7 are the so-called maximally modulated Calderón-Zygmund (maximal) operators. Following the presentation in [24], we recall that given a standard Calderón-Zygmund operator T with kernel K and a family $\Phi=\{\phi_a\}_{a\in A}$ of measurable real-valued functions indexed by an arbitrary set A, we can define the maximally modulated T with respect to Φ:

$$T^\Phi f(x)=\sup_{a\in A}|T(e^{2\pi i\phi_a(\cdot)}f)(x)|.$$

This definition is motivated by the Carleson operator, for which T is the Hilbert transform and Φ is given by $\phi_a(y)=ay$, for every $a\in\mathbb{R}$. Also, mimicking what we do with singular integrals, we can define the maximal version of T^Φ by

$$T^\Phi_*f(x)=\sup_{\varepsilon>0}\sup_{a\in A}|T_\varepsilon(e^{2\pi i\phi_a(\cdot)}f)(x)|,$$

where T_ε is the truncated operator defined by

$$T_\varepsilon f(x)=\int_{|x-y|>\varepsilon}K(x,y)f(y)dy.$$

In the proof of the main result in [24], if we keep track of the constants, the authors show the following:

Theorem 4.8. *Let T be a Calderón-Zygmund operator and let Φ be a family of measurable real-valued functions. Take $\mathcal{T} \in \{T^\Phi, T_*^\Phi\}$. Assume that $\mathcal{T}$ maps $L^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ for $p > 1$ with norm $\|\mathcal{T}\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)}$. Then, $\mathcal{T}$ is under the hypotheses of Theorem 4.7 with*

$$\psi\left(\frac{1}{p-1}\right) = C\|\mathcal{T}\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)}.$$

Alternatively, if we have an estimate on characteristic functions $\|\mathcal{T}\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \le C_p|E|^{1/p}$, then $\mathcal{T}$ is under the hypotheses of Theorem 4.7 with

$$\psi_r\left(\frac{1}{p-1}\right) = C_p.$$

Remark 4.9. *Since T is a Calderón-Zygmund operator, we know that it satisfies Cotlar's inequality and one can readily show that $T_*^\Phi f(x) \lesssim Mf(x) + M(T^\Phi f)(x)$. With this, together with the bounds $\|M\|_{L^{p,\infty}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R})} \lesssim \frac{1}{p-1}$ and $\|M\|_{L^{p,\infty}(\mathbb{R}^n)\to L^p(\mathbb{R})} \le C$ for $p > 1$, we get that*

$$\|T_*^\Phi\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} \lesssim \frac{\|T^\Phi\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)}}{p-1}.$$

Therefore, one can always write a good-λ inequality for T_*^Φ in terms of $\|T^\Phi\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)}$.

The combination of Theorems 4.7 and 4.8 is similar to the results presented in [19], where the authors study weighted strong-type (p,p) estimates for maximally modulated singular integrals T^Φ that satisfy an *a priori* weak-type (p,p) inequality without weights. In the same paper, the authors also show that, for $1 < p \le 2$,

$$\|\mathcal{C}\|_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})} \lesssim \frac{\log_2\left(\frac{1}{p-1}\right)}{p-1}, \tag{4.2}$$

and

$$\|\mathcal{C}_{\mathrm{lac}}\|_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})} \lesssim \log_1\left(\frac{1}{p-1}\right). \tag{4.3}$$

Here $\mathcal{C}_{\mathrm{lac}}$ is the lacunary version of $\mathcal{C}$, defined as a maximally modulated Hilbert transform H^Φ with $\Phi = \{ay\}_{a\in A}$, where the index set $A \subseteq \mathbb{R}$ is lacunary, in the sense that

$$\inf_{a\neq a'\in A} \frac{|a-a'|}{|a|} = C > 0.$$

As we pointed out in Remark 4.6, the use of (4.2) to obtain weighted (p,p) results and then extrapolate does not offer an improvement of Theorem 4.5. However, let us see what we can obtain for the lacunary Carleson operator:

Corollary 4.10. *For every $1 < p \le 2$ and $u \in A_1$,*

$$\|\mathcal{C}_{\mathrm{lac}}\|_{L^p(u)\to L^{p,\infty}(u)} \le C_u \log_1\left(\frac{1}{p-1}\right).$$

In particular,

$$\mathcal{C}_{\mathrm{lac}} : L\log_2 L \log_4 L(u) \longrightarrow R_1(u).$$

Proof. The weak-type (p,p) estimates come from (4.3) together with Theorems 4.7 and 4.8. The endpoint estimate can be obtained by a suitable modification of Antonov's Theorem 2.6 so that it admits the logarithmic blow-up of the constant. □

In this lacunary case, the current best result for the Carleson operator $\mathcal{C}_{\mathrm{lac}}$ is boundedness on $L\log_2 L\log_4 L(\mathbb{T})$ (see [19, 31]), and in fact, it has been recently showed [32] that this is the largest Lorentz space over $\mathbb{T}$ on which $\mathcal{C}_{\mathrm{lac}}$ can be bounded. Hence, what we obtain is an analogue of this endpoint result on $\mathbb{R}$ and with respect to A_1 weights.

4.4. **The operator A.** In this subsection, we shall deal with the operator

$$Af(x) = \left\| \frac{f(x+y)}{y} \right\|_{L^{1,\infty}(\mathbb{R})}.$$

We show a weighted endpoint result for A in the spirit of Theorem 3.3. In [18], the authors prove that a version of A defined on the probability space $([0,1], dx)$ is bounded on $L\log_2 L([0,1])$, and show that in the scale of Orlicz spaces, $L\log_3 L([0,1])$ would be the best possible result. We will show the following:

Theorem 4.11. *For every $u \in A_1$, it holds that*

$$A : L\log_2 L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u).$$

Proof. Since $A\chi_E \le M\chi_E$, the proof is essentially an application of Theorem 3.3. However, we need to make sure that the lack of sublinearity of A is not a problem. The three properties that will replace the missing hypothesis are:

- A is quasilinear: $A(f+g) \le C(Af + Ag)$,
- A is monotone: $|f| \le |g| \Rightarrow Af \le Ag$,
- A is sublinear on functions with disjoint support.

Now, let us examine the proof of Theorem 3.3. Whenever we use sublinearity on a sum of two functions, the quasilinearity of A suffices. The only step where we need it for an arbitrary number of terms is in (3.5), and here quasilinearity is not enough. To get around this problem, we replace (3.5) by the standard dyadic decomposition

$$f \approx \sum_{k\in\mathbb{Z}} 2^k \chi_{\{2^k < f \le 2^{k+1}\}}. \tag{4.4}$$

Since this is an equivalence instead of an equality, we require monotonicity to pass the operator inside the sum. Once this is done, we use that the pieces of (4.4) have disjoint support and A behaves as if it were sublinear. For the decomposition in (4.4) to be finite we only have to start with bounded functions and at the end, add a density argument. □

5. Appendix

Proof of Lemma 2.20. The proof of this result can be found, for instance, in [5, Theorem 5.3.2], but we need to see how the constant behaves and this is not included in classical books. We will proceed as in [10, Lemma 3.10]. By the real interpolation K-method (see [4, Chapter 5]), we have that

$$T : (L^{q_0,s_0}(u), L^{q_1,s_1}(u))_{\theta,\infty} \longrightarrow (L^{q_0,\infty}(u), L^{q_1,\infty}(u))_{\theta,\infty},$$

with constant less than or equal to $M_0^{1-\theta}M_1^{\theta}$, where

$$(A_1, A_2)_{\theta,\infty} = \left\{ f \in A_1 + A_2 : \sup_{t>0} t^{-\theta} K(t, f; A_1, A_2) < \infty \right\},$$

and

$$K(t, f; A_1, A_2) = \inf \left\{ \|f_0\|_{A_1} + t\|f_1\|_{A_2} : f = f_0 + f_1, f_0 \in A_1, f_1 \in A_2 \right\}.$$

Therefore, it is enough to show that:

(i) $\|f\|_{L^{q,\infty}(u)} \leq 2\|f\|_{(L^{q_0,\infty}(u),L^{q_1,\infty}(u))_{\theta,\infty}}$,
(ii) $\|f\|_{(L^{q_0,s_0}(u),L^{q_1,s_1}(u))_{\theta,\infty}} \leq B\|f\|_{L^{q,\infty}(u)}$.

The proof of (i) goes as follows: define $\gamma := \frac{q_0q_1}{q_1-q_0}$, fix $t > 0$ and let $f = f_0 + f_1$ be a decomposition of f in $L^{q_0,\infty}(u) + L^{q_1,\infty}(u)$. Then,

$$\begin{aligned}\sup_{y\leq t^{\gamma}} y^{1/q_0} f_u^*(y) &\leq \sup_{y\leq t^{\gamma}} y^{1/q_0} \left((f_0)_u^* \left(\frac{y}{2}\right) + (f_1)_u^* \left(\frac{y}{2}\right) \right) \\ &\leq \sup_{y\leq t^{\gamma}} 2^{1/q_0} \|f_0\|_{L^{q_0,\infty}(u)} + y^{\frac{1}{q_0}-\frac{1}{q_1}} 2^{1/q_1} \|f_1\|_{L^{q_1,\infty}(u)} \\ &\leq 2(\|f_0\|_{L^{q_0,\infty}(u)} + t\|f_1\|_{L^{q_1,\infty}(u)}).\end{aligned}$$

Taking infimum over all possible decompositions of f, we conclude that

$$\sup_{y\leq t^{\gamma}} y^{1/q_0} f_u^*(y) \leq 2K(t, f; L^{q_0,\infty}(u), L^{q_1,\infty}(u)),$$

and with this estimate,

$$\begin{aligned}2\|f\|_{(L^{q_0,\infty}(u),L^{q_1,\infty}(u))_{\theta,\infty}} &= \sup_{t>0} 2t^{-\theta} K(t, f; L^{q_0,\infty}(u), L^{q_1,\infty}(u)) \\ &\geq \sup_{t>0} \sup_{y\leq t^{\gamma}} t^{-\theta} y^{1/q_0} f_u^*(y) = \sup_{y>0} y^{1/q_0} f_u^*(y) \sup_{t\geq y^{1/\gamma}} t^{-\theta} \\ &= \sup_{y>0} y^{\frac{-\theta}{\gamma}+\frac{1}{q_0}} f_u^*(y) = \|f\|_{L^{q,\infty}(u)}.\end{aligned}$$

For (ii), let $f \in L^{q,\infty}(u)$ and γ as before. For every $t > 0$, we write $f = f_0 + f_1$ with

$$f_0 = f\chi_{\{|f|>f_u^*(t^{\gamma})\}} \quad \text{and} \quad f_1 = f\chi_{\{|f|\leq f_u^*(t^{\gamma})\}}.$$

Now,

$$\begin{aligned}\|f_0\|_{L^{q_0,s_0}(u)} &\leq \left(\int_0^{t^{\gamma}} (f_u^*(y)y^{1/q})^{s_0} y^{\frac{s_0}{q_0}-\frac{s_0}{q}-1} dy \right)^{1/s_0} \leq \|f\|_{L^{q,\infty}(u)} \frac{t^{\gamma\left(\frac{1}{q_0}-\frac{1}{q}\right)}}{\left(\frac{s_0}{q_0}-\frac{s_0}{q}\right)^{1/s_0}} \\ &= t^{\theta} \left(\frac{q_0q}{s_0(q-q_0)} \right)^{1/s_0} \|f\|_{L^{q,\infty}(u)},\end{aligned}$$

by the definition of γ and $\theta = \frac{q_0q_1-qq_1}{qq_0-qq_1}$. Also,

$$\|f_1\|_{L^{q_1,s_1}(u)} \leq f_u^*(t^{\gamma}) \left(\int_0^{t^{\gamma}} y^{\frac{s_1}{q_1}-1} dy \right)^{1/s_1} + \left(\int_{t^{\gamma}}^{\infty} f_u^*(y)^{s_1} y^{\frac{s_1}{q_1}-1} dy \right)^{1/s_1}.$$

For the first term, we multiply and divide by $t^{\gamma/q}$, compute the integral and the bound we get is

$$t^{\theta-1}\left(\frac{q_1}{s_1}\right)^{1/s_1}\|f\|_{L^{q,\infty}(u)}.$$

For the second term, we proceed exactly as for $\|f_0\|_{L^{q_0,s_0}(u)}$ and control it by

$$t^{\theta-1}\left(\frac{q_1 q}{s_1(q_1-q)}\right)^{1/s_1}\|f\|_{L^{q,\infty}(u)}.$$

Bringing the estimates together, we conclude that

$$\begin{aligned}\|f\|_{(L^{q_0,s_0}(u),L^{q_1,s_1}(u))_{\theta,\infty}} &= \sup_{t>0} t^{-\theta}K(t,f;L^{q_0,s_0}(u),L^{q_1,s_1}(u))\\ &\leq \sup_{t>0} t^{-\theta}(\|f_0\|_{L^{q_0,s_0}}(u)+t\|f_1\|_{L^{q_1,s_1}}(u)) \leq B\|f\|_{L^{q,\infty}(u)}.\end{aligned}$$

□

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