POINTWISE BOUNDS FOR INTEGRAL TYPE OPERATORS

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Abstract. The purpose of this paper is to study pointwise upper bounds for integral operators whenever some bound for the input function is known. Applications to the cases of Hardy, Riemann-Liouville or Volterra type operators and the Abel transform, among others, are given. The underlying techniques are closely related to Yano's extrapolation theory.

1. Introduction and motivation

In 1917, Radon [12] found a way of reconstructing a function from its projections, and in 1972 G. Hounsfield was able to build the first x-ray computed tomography scanner, which used the Radon transform to recover an object from its projection data [9]. The case when the object was cylindrically symmetric was originally solved by Abel in 1826 [1], using the nowadays called Abel transform:

$$Af(x) = \int_x^\infty \frac{f(t)t}{\sqrt{t^2 - x^2}} dt. \tag{1.1}$$

This is the special case of the Radon transform in which all projections are identical and hence, a single projection is enough for an exact object reconstruction.

In many papers dealing with the Abel transform, the starting condition on the function f is that "it decays at infinity faster than $1/t$". Obviously, if the information that we have on the function f is just that $f(t) \leq \frac{C}{t}$ for some constant $C > 0$, then we cannot say anything about Af since $A\left(\frac{1}{t}\right) \equiv \infty$. However, if we assume that the decay of f at infinity is a little faster, namely, that there exists $p_0 > 1$ such that, for every $1 < p \leq p_0$ and every $t > 0$,

$$f(t) \leq \frac{C}{t^{2-\frac{1}{p}}}, \tag{1.2}$$

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then $Af(x) < \infty$ for every $x > 0$ and

$$Af(x) \leq C \int_x^\infty \frac{1}{\sqrt{t^2 - x^2} t^{1-\frac{1}{p}}} dt \leq \frac{C x^{\frac{1}{p}-1}}{p-1}.$$

Therefore, taking infimum over $1 < p \leq p_0$, we get that, for every $x > 0$,

$$Af(x) \leq C\Big(1 + \log_+ \frac{1}{x}\Big).$$

The purpose of this paper is to prove that we can obtain the same upper bound for $Af(x)$, under a condition on the decay of f at infinity weaker than (1.2).

This problem seems to be of interest even when we are dealing with integral operators of the form

$$T_K f(x) = \int_0^\infty K(x,t) f(t) dt, \tag{1.3}$$

with K a positive kernel. This class of operators includes

$$S_a f(t) = \int_0^\infty a(s) f(st) ds, \tag{1.4}$$

with a being a positive, locally integrable function. These operators were first introduced by Braverman [4] and Lai [10] and also studied by Andersen in [2]. In particular, they cover the cases of Hardy operators, Riemann-Liouville, Calderón operator, Laplace and Abel transforms, among many others.

Our general setting will be the following: let w be a positive, locally integrable function and set $W(t) = \int_0^t w(s) ds$. We will assume that $W(t) > 0$, for every $t > 0$. Moreover, since W is increasing, it is equivalent to a strictly increasing function and hence, we can assume without loss of generality that W has an inverse, that we will denote by:

$$W^{(-1)} : (0, W(\infty)) \longrightarrow (0, \infty).$$

Let us consider positive, measurable functions f satisfying

$$f(t) \leq \frac{C}{W(t)}, \quad t \in (0, \infty),$$

and an operator T_K as in (1.3). Obviously, for such an f, it holds that

$$T_K f(x) \leq C \int_0^\infty \frac{K(x,t)}{W(t)} dt = M(x),$$

and hence the function M is an upper pointwise bound for T_K on that set of functions. However, on many occasions, $M \equiv \infty$ and no interesting information can be obtained without assuming some extra condition. As in the example of the Abel transform, we will assume that $M \equiv \infty$ but, for every $1 < p \leq p_0$,

$$\int_0^\infty \frac{K(x,t)}{W(t)^{1/p}} dt < \infty.$$

In fact, we will need to have some control on how this quantity blows up when p is close to 1, so to be precise, we will assume that it can be controlled by $\frac{1}{(p-1)^m}$. That is, there exists $m > 0$ such that, for every x,

$$U(x) := \sup_{1<p\leq p_0} (p-1)^{mp} \left(\int_0^\infty \frac{K(x,t)}{W(t)^{1/p}} dt \right)^p < \infty. \tag{1.5}$$

In Section 3, we will see that this is the case of many other interesting examples.

Since our goal is to find pointwise upper bounds, we will work with the following normed spaces:

Definition 1.1. *We say that a measurable function $f \in B(W)$ if and only if W^{-1} is a pointwise upper bound for f, that is*

$$B(W) := \left\{ f \text{ measurable } : \|f\|_{B(W)} = \sup_{t>0} f(t)W(t) < \infty \right\}.$$

We observe that if (1.5) is satisfied, then clearly, there exist positive constants $C_w, C'_w > 0$ such that

$$\int_1^\infty \frac{K(x,t)}{W(t)} dt \leq C'_w \inf_{1<p\leq p_0} \frac{U(x)^{1/p}}{(p-1)^m} \leq C_w U(x) \Big(1 + \log_+ \frac{1}{U(x)} \Big)^m,$$

but this computation fails completely whenever we are dealing with values of the variable t close to zero. Hence, we want to find conditions on the functions $f \in B(W)$ so that the above bound remains true for the whole operator, that is

$$T_K f(x) \leq C_w U(x) \Big(1 + \log_+ \frac{1}{U(x)} \Big)^m.$$

For convenience, we will write $\log_1 x = 1 + \log_+ x$ and we consider the space

$$L(\log L)^m(w) = \left\{ f : \|f\|_{L(\log L)^m(w)} < \infty \right\},$$

where

$$\|f\|_{L(\log L)^m(w)} = \int_0^\infty f^*_w(t) \Big(\log_1 \frac{1}{t} \Big)^m dt.$$

Here, f^*_w is the decreasing rearrangement with respect to w defined by

$$f^*_w(t) = \inf\{s > 0 : w(\{x : |f(x)| > s\}) \leq t\}.$$

As usual, the symbol $f \lesssim g$ will indicate the existence of a constant $C > 0$, independent of all parameters involved, so that $f \leq Cg$. When both $f \lesssim g$ and $g \lesssim f$, we will write $f \approx g$. We emphasize that our constants C may depend on w, since w will always be fixed and is not considered a parameter.

2. Main Results

In order to give the proof of our main theorem, we first need the following result.

Proposition 2.1. *If T is a sublinear operator such that*

$$T : L^p(w) \longrightarrow B(U^{-1/p})$$

is bounded, for every $1 < p \leq p_0$ with constant less than or equal to $\frac{1}{(p-1)^m}$, then

$$T : L(\log L)^m(w) \longrightarrow B(U_m^{-1})$$

is bounded with

$$U_m(t) = U(t)\Big(\log_1 \frac{1}{U(t)}\Big)^m. \tag{2.1}$$

Proof. The proof follows the standard scheme of Yano's extrapolation theorem in its modern version (see [5, 6, 14]) but we include it for the sake of completeness. Let f be a positive function satisfying $\|f\|_\infty \leq 1$. Then,

$$\sup_{t>0} Tf(t)U^{-1/p}(t) \lesssim \frac{\|f\|_{L^p(w)}}{(p-1)^m} \leq \frac{\|f\|_{L^1(w)}^{1/p}}{(p-1)^m},$$

and hence

$$\begin{aligned} Tf(t) &\lesssim \inf_{1<p\leq p_0} \frac{1}{(p-1)^m} \left(\|f\|_{L^1(w)}U(t)\right)^{1/p} \\ &\lesssim \|f\|_{L^1(w)}U(t) \left(\log_1 \frac{1}{\|f\|_{L^1(w)}U(t)}\right)^m \\ &\lesssim \|f\|_{L^1(w)}\Big(\log_1 \frac{1}{\|f\|_{L^1(w)}}\Big)^m U_m(t). \end{aligned}$$

From here, it follows that, if $\|f\|_\infty \leq 1$, then

$$\|Tf\|_{B(U_m^{-1})} \lesssim D_m(\|f\|_{L^1(w)}), \tag{2.2}$$

where $D_m(s) = s\left(\log_1 \frac{1}{s}\right)^m$. Now, for a bounded function with $|f| \geq 1$, whenever $f \neq 0$, we can decompose $f = \sum_{n\geq 0} 2^{n+1} f_n$, where $f_n = 2^{-(n+1)} f \chi_{E_n}$ and $E_n = \{2^n < f \leq 2^{n+1}\}$. If we consider the distribution function

$$\lambda_f^w(y) = \int_{\{x:|f(x)|>y\}} w(x)dx,$$

we have that $\|f_n\|_{L^1(w)} \leq \lambda_f^w(2^n)$. With this estimate, together with the fact that $\|f_n\|_\infty \leq 1$ and $B(U_m^{-1})$ is a normed space, we can use (2.2) on every f_n to conclude that

$$\begin{aligned}\|Tf\|_{B(U_m^{-1})} &\lesssim \sum_{n=0}^{\infty} 2^n D_m(\|f_n\|_{L^1(w)}) \lesssim \sum_{n=0}^{\infty} 2^n D_m(\lambda_f^w(2^n)) \\ &\lesssim \int_0^{\infty} D_m(\lambda_f^w(y))dy = \|f\|_{L(\log L)^m(w)},\end{aligned}$$

as we wanted to see. We extend this estimate to a general function (not necessarily bounded) by a density argument. □

Remark 2.2. *It is easy to see that, if f is a decreasing function, then*

$$\|f\|_{L(\log L)^m(w)} \approx \int_0^{\infty} f(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt.$$

The following result follows immediately by Hölder's inequality:

Lemma 2.3. *Let w be a positive, locally integrable function on $(0,\infty)$ and let P_w be the generalized Hardy operator*

$$P_w f(x) = \frac{1}{W(x)} \int_0^x f(s)w(s)ds.$$

Then,

$$P_w : L^p(w) \longrightarrow B(W^{1/p})$$

is bounded, with constant 1.

Now, we are ready to prove the main result of this paper, following the ideas introduced in [7]:

Theorem 2.4. *Let T_K be defined as in* (1.3) *and satisfying* (1.5)*. U_m will stand for the expression in* (2.1)*. Then, for every x,*

$$T_K f(x) \lesssim \|f\|_{D_m(W)} U_m(x),$$

where

$$\begin{aligned}\|f\|_{D_m(W)} &= \|f\|_{B(W)} \\ &\quad + \int_0^1 \sup_{s>0} \Big(\min\Big(\frac{W(s)}{W(t)}, 1\Big) f(s)\Big) \Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt.\end{aligned}$$

Proof. By (1.5), we have that

$$T_K : B(W^{1/p}) \longrightarrow B(U^{-1/p})$$

with constant less than or equal to $(p-1)^{-m}$ and hence, by the previous lemma,

$$T_K \circ P_w : L^p(w) \longrightarrow B(U^{-1/p}),$$

is bounded, with the same behavior of the constant. Then, applying Proposition 2.1, we obtain that

$$T_K \circ P_w : L(\log L)^m(w) \longrightarrow B(U_m^{-1})$$

is bounded. Now, since for t small enough, $t \le \delta < 1$,

$$\Big(\log_1 \frac{1}{W(t)}\Big)^m \approx \int_t^1 \Big(\log_1 \frac{1}{W(s)}\Big)^{m-1} \frac{w(s)}{W(s)} ds,$$

we have that

$$\int_0^\delta g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt \lesssim \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt.$$

Therefore, by Remark 2.2, if g is decreasing,

$$\begin{aligned} \sup_{t>0} \frac{T_K(P_w g)(t)}{U_m(t)} &\lesssim \int_0^\infty g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt \\ &\lesssim \|g\|_{L^1(w)} + \int_0^\delta g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt \\ &\lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt. \end{aligned} \tag{2.3}$$

Let us now assume that $f \in B(W)$ is a decreasing function satisfying

$$\int_0^1 \frac{\sup_{s\le t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt < \infty.$$

Set $H(t) = \sup_{s\le t} W(s)f(s)$. With this definition, H is an increasing function such that $H(0) = 0$ and $\frac{H(t)}{W(t)}$ is decreasing, so we have that $H\left(W^{(-1)}(t)\right)$ is quasi-concave on $(0, W(\infty))$. It is known (see [3, Chapter 2]) that in this case, there exists h decreasing such that $H\left(W^{(-1)}(t)\right) \approx \int_0^t h(s)ds$, with equivalence constant 2, so by a change of variables, there exists g decreasing such that

$$\frac{1}{2} H(t) \le \int_0^t g(s)w(s)ds \le 2H(t).$$

On the other hand,

$$f(t) \le \frac{H(t)}{W(t)} \approx \frac{\int_0^t g(s)w(s)ds}{W(t)} = P_w g(t),$$

and thus

$$T_K f(t) \lesssim T_K(P_w g)(t).$$

Therefore, using (2.3)

$$\begin{aligned} \sup_{t>0} \frac{T_K f(t)}{U_m(t)} &\lesssim \sup_{t>0} \frac{T_K(P_w g)(t)}{U_m(t)} \\ &\lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt. \end{aligned}$$

Since

$$\|P_w g\|_{B(W)} = \sup_{t>0} W(t) \frac{\int_0^t g(s)w(s)ds}{W(t)} \approx \sup_{t>0} H(t) = \|f\|_{B(W)},$$

and

$$P_w g(t) \approx \frac{\sup_{s\le t} W(s)f(s)}{W(t)},$$

we obtain that, for every decreasing function $f \in B(W)$,

$$\sup_{t>0} \frac{T_K f(t)}{U_m(t)} \lesssim \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\le t} W(s)f(s)}{W(t)} \Big(\log_1 \frac{1}{W(t)} \Big)^{m-1} w(t)dt. \tag{2.4}$$

Finally, if we take a general function $f \in B(W)$, we can consider its least decreasing majorant $F(t) = \sup_{r\ge t} f(r)$. We have that $F \in B(W)$ is decreasing and $f \le F$. Hence, $T_K f(x) \le T_K F(x)$ and the result follows immediately applying (2.4) to the function F, since

$$\|F\|_{B(W)} = \sup_{t>0} F(t)W(t) = \sup_{t>0} \sup_{r\ge t} f(r)W(t) = \sup_{t>0} f(t)W(t) = \|f\|_{B(W)},$$

and

$$\begin{aligned} \frac{\sup_{s\le t} W(s)F(s)}{W(t)} &= \frac{\sup_{s\le t} W(s) \sup_{r\ge s} f(r)}{W(t)} = \frac{\sup_{r>0} f(r)W(\min(t,r))}{W(t)} \\ &= \frac{\max\big(\sup_{s\le t} f(s)W(s), W(t)\sup_{s\ge t} f(t)\big)}{W(t)} \\ &= \sup_{s>0} \left(\min\left(\frac{W(s)}{W(t)}, 1 \right) f(s) \right). \end{aligned}$$

□

Notice that the natural setting for Theorem 2.4 is that of decreasing functions, and we just extend it to general functions by considering their least decreasing majorants. In fact, if f is itself decreasing, the expression for $\|f\|_{D_m(W)}$ can be written in a simpler way. The next corollary is just the result that we get in this setting and corresponds to the estimate in (2.4):

Corollary 2.5. *Under the hypotheses of Theorem 2.4 we have that, for every decreasing function f,*

$$T_K f(x) \lesssim \|f\|_{D_m(W)} U_m(x),$$

where

$$\|f\|_{D_m(W)} = \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\le t} W(s)f(s)}{W(t)} \Big(\log_1 \frac{1}{W(t)} \Big)^{m-1} w(t)dt.$$

Finally, the following corollary gives a bound for the iterative operator of order $n \in \mathbb{N}$, $T_K^n f = T_K(T_K^{n-1} f)$:

Corollary 2.6. *Assume that T_K satisfies* (1.5)*, with $U \approx W^{-1}$. Then, for every $n \in \mathbb{N}$, we have that*

$$T_K^n f(x) \lesssim \|f\|_{D_{nm}(W)} \frac{1}{W(x)} \left(\log_1 W(x)\right)^{nm}.$$

Proof. Since T_K satisfies (1.5), with $U \approx W^{-1}$, we have that

$$T_K : B(W^{1/p}) \longrightarrow B(W^{1/p}),$$

with constant less than or equal to $(p-1)^{-m}$, so we can iterate to conclude that the same holds for T_K^n, with constant controlled by $(p-1)^{-nm}$. The proof now follows as in Theorem 2.4. □

3. Examples and applications

In this section, we will use Theorem 2.4 on some interesting examples. Obviously, if one is only interested in decreasing functions, all the conditions can be written as in Corollary 2.5 instead.

(I) The Abel transform: Let us start by solving the initial question about the Abel transform.

Corollary 3.1. *If a positive measurable function $f(t) \lesssim 1/t$ satisfies that*

$$\int_1^\infty \sup_y \left(f(y) y \min(y,t)\right) \frac{dt}{t^2} < \infty \tag{3.1}$$

then, for every $x > 0$,

$$Af(x) \lesssim \log_1 \frac{1}{x}.$$

Remark 3.2. Before giving the proof, we should emphasize the fact that it is very easy to verify that condition (3.1) is weaker than (1.2).

Proof. First of all, making a change of variables, it is immediate to see that, if $g(s) = f(\frac{1}{s})\frac{1}{s^2}$ and

$$T_K g(x) = \int_0^x \frac{g(s)}{\sqrt{x^2 - s^2}} ds,$$

then, for every $x > 0$,

$$Af(x) = \frac{1}{x} T_K g\Big(\frac{1}{x}\Big). \tag{3.2}$$

On the other hand, we have that

$$\sup_{1<p\le 2} (p-1)^p \left(\int_0^x \frac{1}{\sqrt{x^2 - s^2} s^{1/p}} ds \right)^p \approx \frac{1}{x} < \infty,$$

and therefore, applying Theorem 2.4, we get

$$T_K g(x) \lesssim \frac{\log_1 x}{x},$$

whenever $g \in B(W)$ with $W(t) = t$ and

$$\int_0^1 \sup_{s>0} \left(g(s) \min\left(\frac{s}{t}, 1\right)\right) dt < \infty.$$

The result now follows rewriting this condition in terms of f and using (3.2). □

(II) The Riemann-Liouville operator: Given $\alpha > 0$, let us consider the operator

$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1} dt.$$

Then, making the change of variables $y = \frac{t}{x}$, we have that

$$R_\alpha f(x) = x^\alpha \int_0^1 (1-y)^{\alpha-1} f(yx) dy := x^\alpha I_\alpha f(x),$$

and hence

$$\sup_{1<p\leq 2} (p-1)^p \left(I_\alpha\Big(\frac{1}{y^{1/p}}\Big)(x)\right)^p \lesssim \frac{1}{x}.$$

Consequently, if we take $W(t) = t$ and $U(t) = \frac{1}{t}$, we have that I_α is under the hypotheses of Theorem 2.4 and therefore

$$I_\alpha f(x) \lesssim \frac{\log_1 x}{x},$$

whenever $f(t) \lesssim 1/t$ and satisfies that

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{s}{t}, 1\right) f(s)\right) dt < \infty.$$

Hence, under these conditions on f, it holds that, for every $x > 0$,

$$R_\alpha f(x) \lesssim x^{\alpha-1}(\log_1 x).$$

(III) Iterative operators: Observe that in the two previous examples, the function U coincides with W^{-1}, and hence we can apply Corollary 2.6 to obtain the following:

(III.1) For every $n \in \mathbb{N}$ and every positive measurable function $f(t) \lesssim 1/t$ such that

$$\int_1^\infty \sup_y \left(f(y) y \min(y,t)\right) (\log_1 t)^{n-1} \frac{dt}{t^2} < \infty,$$

it holds that, for every $x > 0$,

$$A^n f(x) \lesssim \left(\log_1 \frac{1}{x}\right)^n.$$

(III.2) For every $n \in \mathbb{N}$ and every positive measurable function $f(t) \lesssim 1/t$ such that

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{s}{t},1\right) f(s)\right) \left(\log_1 \frac{1}{t}\right)^{n-1} dt < \infty,$$

it holds that, for every $x > 0$,

$$R_\alpha^n f(x) \lesssim x^{\alpha-1} (\log_1 x)^n.$$

(IV) Braverman-Lai's operators: Let us now consider the operator S_a defined in (1.4) and let us assume the following: there exist an increasing function $D \geq 0$, with $D(t) = 0$ if and only if $t = 0$, and a function E so that, for some $m > 0$ and every $1 < p \leq p_0$,

$$\int_0^\infty \left(\sup_{t>0} \frac{E(t)}{D(st)}\right)^{1/p} a(s) ds \lesssim \frac{1}{(p-1)^m}. \tag{3.3}$$

Then, one can immediately see that, for every $t > 0$,

$$\int_0^\infty \frac{a(s)}{D(st)^{1/p}} ds \lesssim \frac{1}{(p-1)^m E(t)^{1/p}},$$

and hence, (1.5) holds with $W = D$ and $U \lesssim E^{-1}$. A direct consequence of Theorem 2.4 is the following:

Corollary 3.3. *If* (3.3) *holds, then, for every* $f \in B(D)$ *satisfying*

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{D(s)}{D(t)},1\right) f(s)\right) \left(\log_1 \frac{1}{D(t)}\right)^{m-1} dD(t) < \infty,$$

we have that

$$S_a f(x) \lesssim \frac{1}{E(x)} (\log_1 E(x))^m.$$

Remark 3.4. In the simplest case, when $a(s) = \chi_{(0,1)}(s)$, the operator $S_a f(t) = Sf(t) = \frac{1}{t}\int_0^t f(s)ds$ is the Hardy operator and we obtain that if $D(t) = E(t) = t$, we can take $m = 1$ to conclude that

$$S_a f(t) \lesssim \frac{\log_1 t}{t},$$

whenever $f(t) \lesssim 1/t$ and

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{s}{t},1\right) f(s)\right) dt < \infty.$$

By taking a function f such that $f(t) = \frac{1}{t}$, whenever $t > 1$, we see that the pointwise bound cannot be improved. However, in this particular example, in order to get that pointwise bound, it is possible to weaken the condition on the function near 0 by simply assuming that $f \in L^1(0,1)$.

(V) Other applications: In Theorem 2.4, the condition that we require on f is that its least decreasing majorant F satisfies $\|F\|_{D_m(W)} < \infty$. To finish this section, we will present two more versions of our main result in which the role of F is played by the decreasing rearrangement f^* and the level function f°, respectively.

Assume that $K(x,t)$ is decreasing in t. Then, by Hardy's inequality [3, Theorem 2.2], we have that, for every function f,

$$T_K f(x) = \int_0^\infty K(x,t)f(t)dt \leq \int_0^\infty K(x,t)f^*(t)dt = T_K(f^*)(x),$$

so we can apply Corollary 2.5 to f^* and write the following result:

Corollary 3.5. *Under the hypotheses of Theorem 2.4 if, for every $x > 0$, $K(x,t)$ is decreasing in $t \in (0,\infty)$, then*

$$T_K f(x) \lesssim \|f^*\|_{D_m(W)} U_m(x).$$

Similarly, assume now that we have a Volterra operator

$$V_K f(x) = \int_0^x K(x,t)f(t)dt,$$

with $K(x,t)$ decreasing in $t \in (0,x)$. In [11], the authors show that, for every bounded function $f \geq 0$ with compact support in $(0,\infty)$, it holds that

$$V_K f(x) \leq V_K(f^\circ)(x),$$

where f° is a decreasing function associated with f called the Halperin level function (see [8, 13]). Therefore, this estimate together with Corollary 2.5 and Fatou's lemma yield:

Corollary 3.6. *Under the hypotheses of Theorem 2.4, if $K(x,t)$ is decreasing in $t \in (0,x)$ for every $x > 0$, then*

$$V_K f(x) \lesssim \|f^\circ\|_{D_m(W)} U_m(x).$$

4. Generalization to sublinear operators

Although our motivation has been to study integral operators with positive kernels, our main result can be extended to more general operators as follows:

Theorem 4.1. *Let T be a sublinear operator such that, for every x*

$$U(x) := \sup_{1<p\leq p_0} \sup_{\|f\|_{B(W^{1/p})}\leq 1} (p-1)^{pm} Tf(x)^p < \infty.$$

Then, we have that

$$Tf(x) \lesssim \|f\|_{D_m(W)} U_m(x).$$

In the proof of Theorem 2.4, we make use of the fact that the operators T_K are monotone. Since now we do not have this property on T, we will need to introduce auxiliary functions κ and ρ to get around this problem.

Proof. We will follow the proof of Theorem 2.4. Let κ be an arbitrary function with $\|\kappa\|_\infty \leq 2$. Define

$$T_\kappa f := T(\kappa f).$$

By our assumption, it is easy to check that, for every $1 < p \leq p_0$,

$$T_\kappa : B(W^{1/p}) \longrightarrow B(U^{-1/p}),$$

with constant controlled by $(p-1)^{-m}$. As before, we get that, for every function $\|\kappa\|_\infty \leq 2$ and every g decreasing,

$$\sup_{t>0} \frac{T_\kappa(P_w g)(t)}{U_m(t)} \lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt. \tag{4.1}$$

Let us now assume that $f \in B(W)$ is a decreasing function satisfying that

$$\int_0^1 \frac{\sup_{s\leq t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt < \infty.$$

If $H(t) = \sup_{s\leq t} W(s)f(s)$, we have the existence of a decreasing function g such that

$$\frac{1}{2}H(t) \leq \int_0^t g(s)w(s)ds \leq 2H(t).$$

With this,

$$f(t) \leq \frac{H(t)}{W(t)} \leq \frac{2\int_0^t g(s)w(s)ds}{W(t)} = 2P_w g(t),$$

so we can write, for some $\|\kappa\|_\infty \leq 2$,

$$f(t) = \kappa(t)P_w g(t).$$

Therefore, for every function ρ with $\|\rho\|_\infty \leq 1$, we can use (4.1) with $\|\kappa\rho\|_\infty \leq 2$ to show that

$$\begin{aligned} \sup_{t>0}\frac{T(\rho f)(t)}{U_m(t)} &= \sup_{t>0}\frac{T_{\kappa\rho}(P_w g)(t)}{U_m(t)} \\ &\lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt \\ &\approx \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\leq t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt. \end{aligned} \tag{4.2}$$

Choosing $\rho \equiv 1$, we finish the proof in the decreasing case. For a general function $f \in B(W)$, we consider its least decreasing majorant $F(t) = \sup_{r\geq t} f(r)$, which lies in $B(W)$ and satisfies $f \leq F$. Hence, we write $Tf(x) = T(\rho F)(x)$ for some $\|\rho\|_\infty \leq 1$, and the result follows immediately applying (4.2) together with

$$\|F\|_{B(W)} = \|f\|_{B(W)}$$

and

$$\frac{\sup_{s\leq t} W(s)F(s)}{W(t)} = \sup_{s>0}\left(\min\Big(\frac{W(s)}{W(t)}, 1\Big) f(s)\right).$$

□

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