Some bifurcations related to homoclinic tangencies for 1-parameter families of symplectic diffeomorphisms*

J. C. Tatjer

Dept. Matemàtica Aplicada i Anàlisi. Universitat de Barcelona.
Gran Via 585. 08071 Barcelona. Spain.

1 Introduction

The goal of this paper is the generalization of the following result to symplectic diffeomorphisms (see [3, 4] for more details):

Let $\{f_a\}_{a\in V}$ be a smooth family of planar diffeomorphisms such that $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is smooth enough with respect to the variables and the parameter and V is an open neighbourhood of a_0. Suppose that for $a = a_0$, f_{a_0} has a dissipative saddle fixed point p_0 with a non-degenerate homoclinic tangency of its invariant manifolds, which unfolds generically with $\{f_a\}_{a\in V}$. Then, under generic assumptions and for n large enough, there exist values of the parameter a_n^+ and a_n^- such that a) $f_{a_n^+}$ has an n-periodic saddle-node point p_n^+ and b) $f_{a_n^-}$ has an n-periodic flip point. Moreover the family $\{f_a\}_{a\in V}$ tends, after an n-dependent change of variables and parameter, and near the homoclinic tangency point, to the logistic map in the following form: $g_\epsilon(x, y) = (y, y^2 + \epsilon)$.

We will obtain a similar result for area-preserving maps. But in this case the limit map is the conservative Hénon map, and the bifurcations are pararabolic. For a four dimensional symplectic diffeomorphisms we can obtain also a similar result for the existence of bifurcations, but we only consider the case in which all the eigenvalues of the initial fixed points

*Preprint version. Published in Hamiltonian systems with three or more degrees of freedom (S'Agar, 1995), 595599, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 533, Kluwer Acad. Publ., Dordrecht, 1999.

are real and different. In this last case it is not possible to have a limit diffeomorphism as we will see. The main difference between the conservative and dissipative case is that in the conservative case there is not a smooth change of variables that transforms a map near a hyperbolic fixed point to its linear part. But with the aid of the normal form we can prove our results in a similar way than in the dissipative case.

We will split our results in two sections: one corresponding to 1-parameter families of area preserving maps and the other to 1-parameter families of four dimensional symplectic diffeomorphisms. From now on, when we say smooth, we mean sufficiently differentiable.

2 Families of area preserving maps

Let $a_0 \in \mathbb{R}$ and $V \subset \mathbb{R}$ be a neighbourhood of a_0. Let $\{f_a\}_{a\in V}$ be a smooth family of smooth area preserving maps, $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, such that for $a = a_0$ there exists a hyperbolic fixed point p_0. Then there exists a smooth map $p = p(a)$ such that $p(a_0) = p_0$ and $p(a)$ is a hyperbolic fixed point of f_a. We denote $\lambda = \lambda(a)$ and λ^{-1} the eigenvalues of $p(a)$. It is not restrictive to suppose that $\lambda > 1$ (if not we take f_a^2 instead of f_a).

Let $g_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$g_a \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} t_1 \exp(-\alpha - k(t_1 t_2, a)) \\ t_2 \exp(\alpha + k(t_1 t_2, a)) \end{pmatrix},$$

where $\alpha = \log(\lambda)$ and k is a smooth map such that $k(0) = 0$. Then it is possible to prove, using [1]:

Theorem 2.1 *There exist a C^3 area preserving map $\vec{x} : W \times V_1 \subset \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$ and g_a as before, where W is a neighbourhood of $\{(t_1, t_2) \ : \ t_1 t_2 = 0\}$, and $V_1 \subset V$ is a neighbourhood of a_0, such that:*

1. *$\vec{x}(g_a(t_1, t_2), a) = f_a(\vec{x}(t_1, t_2, a))$.*
2. *$\vec{x}(\cdot, a)$ is a local conjugacy of f_a with g_a.*

We notice that using the previous result we can suppose that the fixed point $p(a)$ is always $(0, 0)$ and the map f_a is equal to g_a in a neighbourhood of $(0, 0)$. Then the map $\vec{x}$ will be the identity map in a neighbourhood of $(0, 0)$. Finally we can also assume that $a_0 = 0$. From now on, we will suppose that our family satisfies these properties.

Now we need to give a definition of homoclinic tangency unfolding generically. Suppose that there is a homoclinic tangency corresponding to the fixed point $(0,0)$ of f_0. It is not restrictive to assume that the point of homoclinic tangency is $p_1 = (\bar{t}_1, 0)$. By the definition of homoclinic point we have that there exists a real number $\bar{t}_2$ such that $\vec{x}(0, \bar{t}_2, 0) = p_1$. Here $\vec{x} = (x, y)$ is the map of theorem 2.1.

Definition 2.2 *We say that the fixed point* $(0,0)$ *of* f_0 *has a a non-degenerate homoclinic tangency which unfolds generically with a if:* $y(0, \bar{t}_2, 0) = 0$, $D_2 y(0, \bar{t}_2, 0) = 0$, $D_{22} y(0, \bar{t}_2, 0) \neq 0$ *and* $D_3 y(0, \bar{t}_2, 0) \neq 0$.

The main result of the section is the following theorem:

Theorem 2.3 *Let* $\{f_a\}_{a \in V}$ *a smooth family of area preserving maps having, for* $a = a_0$*, a non-degenerate homoclinic tangency which unfolds generically with a. Then, for n large enough, there exist values of the parameter* a_n^+, a_n^- *such that:*

1. $f_{a_n^+}$ *has an n-periodic parabolic point (double egeinvalue 1).*
2. $f_{a_n^-}$ *has an n periodic parabolic point with reflection (double eigenvalue -1).*
3. $\lim_{n \to \infty} a_n^{\pm} = a_0$.

Moroever, there exist open sets $U_1 \subset \mathbb{R}$, $U_2 \subset \mathbb{R}^2$ *and maps* $M_n : U_1 \longrightarrow M_n(U_1) \subset \mathbb{R}$, $\psi_{n,\epsilon} : U_2 \longrightarrow \psi_{n,\epsilon}(U_2) \subset \mathbb{R}^2$ *such that:*

(a) M_n *is a diffeomorphism.*

(b) *For each compact set K in the* $(\epsilon, \tilde{t}_1, \tilde{t}_2)$ *space the images of K under the maps* $(\epsilon, \tilde{t}_1, \tilde{t}_2) \longrightarrow (M_n(\epsilon), \psi_{n,\epsilon}(\tilde{t}_1, \tilde{t}_2))$ *converge for* $n \to \infty$ *to* $(0, 1, 0)$.

(c) *The domains of the maps* $(\epsilon, \tilde{t}_1, \tilde{t}_2) \longrightarrow (\epsilon, \psi_{n,\epsilon}^{-1} \circ f^n_{M_n(\epsilon)} \circ \psi_{n,\epsilon})$*, converge, for* $n \to \infty$*, to all of* $\mathbb{R}^3$ *and the maps converge, in the* C^3 *topology to the map:*
$$(\epsilon, \tilde{t}_1, \tilde{t}_2) \mapsto (\epsilon, h_\epsilon(\tilde{t}_1, \tilde{t}_2))$$
with $h_\epsilon(\tilde{t}_1, \tilde{t}_2) = (1 - \tilde{t}_2 - \epsilon \tilde{t}_1^2, \tilde{t}_1)$.

Idea of the proof:

First we define a map $C = C(n, t_1, t_2, a)$ such that $C = \exp[(-\alpha - k(C, a))n]t_1 t_2$. It is possible to prove that such a map exists for n large enough and (t_1, t_2) near $(0, 0)$. Moreover C is diferentiable with respect (t_1, t_2, a). Then we define:

$$s_1 = \exp((-\alpha - k(C, a))n), \quad s_2 = n \exp((-\alpha - k(C, a))n).$$

To find a fixed point of f_a^n with a double eigenvalue 1 is equivalent to solve the system of equations:

$$\begin{array}{rcl} f_a^n(t_1, s_1 t_2) - (t_1, s_1 t_2) & = & 0 \\ \operatorname{tr} D f_a^n(t_1, s_1 t_2) + 2 & = & 0. \end{array}$$

Using the map $\vec{x}$ defined before, we can substitute the first equation by $\vec{x}(s_1 t_1, t_2, a) - (t_1, s_1 t_2) = 0$ and the second equation by another equation depending on (s_1, s_2, t_1, t_2, a). When we put $s_1 = s_2 = 0$ in the two equations, they are satisfied for $(t_1, t_2) = (\bar{t}_1, \bar{t}_2)$ because there is a homoclinic tangency in $(\bar{t}_1, 0) = \vec{x}(0, \bar{t}_2, a_0)$. Moreover, as the homoclinic tangency is non-degenerate and unfolds generically, we can aply the implicit function theorem and obtain functions $a = a(s_1, s_2)$, $t_1 = t_1(s_1, s_2)$, $t_2 = t_2(s_1, s_2)$, such that satisfy the system of equations and $t_1(0, 0) = \bar{t}_1$, $t_2(0, 0) = \bar{t}_2$, $a(0, 0) = a_0$. Taking into acount the definition of s_1 and s_2 we can easily prove the items 1., 2. and 3.

For the second part of the theorem, we can suppose, via a change of variables and parameter, that the map $\vec{x} = (x, y)$ has the following properties: a) The point of homoclinic tangency is $(1, 0)$ for $a = 0$. b) $x(0, 1, a) = 1$, $y(0, 1, a) = a$, $D_2 y(0, 1, a) = 0$, $D_{22} y(0, 1, 0) \neq 0$, $D_1 y(0, 1, 0) D_2 x(0, 1, 0) = -1$, for all a in a neighbourhood of 0.

Let $\beta = D_{22} y(0, 1, 0)$, $\gamma = D_1 y(0, 1, 0)$, $\delta_0 = D_1 x(0, 1, 0)$ $\delta_1 = D_{11} y(0, 1, 0)$, $\delta_2 = D_{12} y(0, 1, 0)$. We consider the following change of variables and parameters for the map f_a with variables (t_1, t_2) :

$$(a, t_1, t_2) = ((1 - \gamma)s_1 + \epsilon s_1^2, 1 + s_1 \tilde{t}_1, s_1 + s_1^2 \tilde{t}_2).$$

It is possible to prove that if we perform this change to the map f_a^n and make n tend to infinity, then we obtain the map:

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} \mapsto \begin{pmatrix} \delta_0 - \gamma^{-1} \tilde{t}_2 \\ \gamma \tilde{t}_1 + \frac{1}{2}\delta_1 + \delta_2 \tilde{t}_2 + \frac{1}{2}\beta \tilde{t}_2^2 + \epsilon \end{pmatrix}$$

It is easy to see that this map is conjugate to the map h_ϵ of the item (c) of the theorem. $\square$

Remark 2.4 *The thesis of the first part of the theorem is also proved, with other tools in [2].*

3 Symplectic maps of dimension four

Let $\{f_a\}_{a\in V}$ as in the beginning of the previous section, but now $f_a : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, and f_a is a symplectic map. We suppose that for $a = a_0$ f_a has a hyperbolic fixed point p_0. As before there exists a smooth map $p = p(a)$ such that $p(a_0) = a_0$ and $p(a)$ is a hyperbolic fixed point of f_a for $a \in V$ (if this is true only in a subset of V we substitute V by the new neighbourhood). We suppose that the eigenvalues of $p(a)$ are $\lambda = \lambda(a) > 1$, $\mu = \mu(a) > 1$ and λ^{-1}, μ^{-1}. Moreover we assume that $\lambda \neq \mu$. Let $G_a : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ a symplectic map such that:

$$G_a \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 \exp((-\alpha_1 - k_1(t_1t_3, t_2t_4, a)) \\ t_2 \exp((-\alpha_2 - k_2(t_1t_3, t_2t_4, a)) \\ t_3 \exp((\alpha_1 + k_1(t_1t_3, t_2t_4, a)) \\ t_4 \exp((\alpha_2 + k_2(t_1t_3, t_2t_4, a)) \end{pmatrix}$$

where $\alpha_1 = \log(\lambda)$, $\alpha_2 = \log(\mu)$, k_1 and k_2 are smooth maps and $k_1(0,0,a) = k_2(0,0,a) = 0$ for all a. Then again it is possible to prove, using [1]:

Theorem 3.1 *Assume that the map f_{a_0} does not have resonaces of low order (as many as we need, see [1] for details), but the unavoidable. Then there exist a C^3 symplectic map $\vec{x} : W \times V_1 \subset \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4$ and G_a as before, where W is a neighbourhood of $\{(t_1,t_2,t_3,t_4) \ : \ t_1 = t_3 = 0$ or $t_2 = t_4 = 0\}$ and $V_1 \subset V$ is a neighbourhood of a_0, such that:*

1. *$\vec{x}(G_a(t_1,t_2,t_3,t_4),a) = f_a(\vec{x}(t_1,t_2,t_3,t_4,a))$.*
2. *$\vec{x}(\cdot,a)$ is a local conjugacy of f_a with G_a.*

As before, we can suppose that the fixed point $p(a)$ is always $(0,0,0,0)$ and the map f_a is equal to G_a in a neighbourhood of $(0,0,0,0)$. Then the map $\vec{x}$ will be the identity in a negihbourhood of $(0,0,0,0)$. We also shall assume that $a_0 = 0$. From now on, we shall suppose that our family satisfies these properties.

Now we need to give a definition of homoclinic tangency unfolding generically. Suppose that there is a homoclinic tangency corresponding to the fixed point $(0,0,0,0)$ of f_{a_0}. It is not restrictive to assume that the point of homoclinic tangency $p_1 = (\bar{t}_1, \bar{t}_2, 0, 0)$. By the definition of homoclinic point we have that there exists $(\bar{t}_3, \bar{t}_4)$ such that $\vec{x}(0,0,\bar{t}_3,\bar{t}_4,0) = p_1$. Here $\vec{x} = (x,y,z,t)$ is the map of theorem 3.1.

Definition 3.2 *We say that the fixed point* $(0,0,0,0)$ *of* f_0 *has a non-degenerate homoclinic tangency in* p_1 *which unfolds generically with* a *if:*

1. $z(0,0,\bar{t}_3,\bar{t}_4,0) = 0$, $t(0,0,\bar{t}_3,\bar{t}_4,0) = 0$. *(existence of intersection)*

2. $$\begin{vmatrix} D_3 z(0,0,\bar{t}_3,\bar{t}_4,0) & D_4 z(0,0,\bar{t}_3,\bar{t}_4,0) \\ D_3 t(0,0,\bar{t}_3,\bar{t}_4,0) & D_4 t(0,0,\bar{t}_3,\bar{t}_4,0) \end{vmatrix} = 0 \quad \textit{(tangential intersection)}$$

3. *Let*
$$g(t_3,t_4,a) = \begin{vmatrix} D_3 z(0,0,t_3,t_4,a) & D_3 t(0,0,t_3,t_4,a) \\ D_4 z(0,0,t_3,t_4,a) & D_4 t(0,0,t_3,t_4,a) \end{vmatrix}$$
Then $\operatorname{rang} D(g(\cdot,\cdot,0), z(0,0,\cdot,\cdot,0), t(0,0,\cdot,\cdot,0))(\bar{t}_3,\bar{t}_4) = 2$. *(nondegenerate tangency).*

4. $\operatorname{rang} D(z(0,0,\cdot,\cdot,\cdot), t(0,0,\cdot,\cdot,\cdot))(\bar{t}_3,\bar{t}_4,0) = 2$ *(good dependence on the parameter).*

The first condition says that there is an intersection between the invariant manifolds, the second that the intersection is tangential, the third that the tangency is nondegenerate and the last one that the tangency unfolds generically with respect to the parameter.

Using this definition it is possible to prove that there exist smooth maps $a = a(r)$, $t_3 = t_3(r)$, $t_4 = t_4(r)$ defined for $r \in \mathbb{R}$ in a neighbourhood of 0, such that $a(0) = 0$, $t_3(0) = \bar{t}_3$, $t_4(0) = \bar{t}_4$, $\dot{a}(0) = 0$, $\ddot{a}(0) \neq 0$, and $f_{a(r)}$ has a homoclinic point in $\vec{x}(0,0,t_3(r),t_4(r),a(r))$. The main result of this section is the following theorem:

Theorem 3.3 *Let* $\{f_a\}_{a\in V}$ *a family of symplectic maps in dimension four with the same hypothesis than in the beginning of the section. We suppose that for* $a = a_0$, f_a *has a non-degenerate homoclinic tangency which unfolds generically with* a. *Then, for* n *large enough, there exist values of the parameter* a_n^+, a_n^- *such that:*

1. *$f_{a_n^+}$ has an n-periodic partially parabolic point (double egeinvalue 1). If the other eigenvalues are $\lambda_n > 1$ and λ_n^{-1} then $\lambda_n \to \infty$ when $n \to \infty$.*

2. *$f_{a_n^-}$ has an n-periodic partially parabolic point with reflection (double eigenvalue -1). The other eigenvalues satisfy the same property of the previous item.*

3. $\lim_{n\to\infty} a_n^{\pm} = a_0$.

Idea of the proof:

We proceed as in the area preserving case. We define maps

$$C_1 = C_1(n, t_1, t_2, t_3, t_4, a) \quad \text{and} \quad C_2 = C_2(n, t_1, t_2, t_3, t_4, a)$$

such that

$$C_1 = \exp[(-\alpha_1 - k_1(C_1, C_2, a))n]t_1 t_3 \quad \text{and} \quad C_2 = \exp[(-\alpha_2 - k_2(C_1, C_2, a))n]t_2 t_4.$$

Then we define:

$$s_1 = \exp((-\alpha_1 - k_1(C_1, C_2, a))n), \quad s_2 = n \exp((-\alpha_1 - k_1(C_1, C_2, a))n),$$

$$s_3 = \exp((-\alpha_2 - k_2(C_1, C_2, a))n), \quad s_4 = n \exp((-\alpha_2 - k_2(C_1, C_2, a))n).$$

To find a fixed point of f_a^n with double eigenvalue 1 is equivalent to solve the system:

$$\begin{aligned} f_a^n(t_1, t_2, s_1 t_3, s_3 t_4) - (t_1, t_2, s_1 t_3, s_3 t_4) &= 0 \\ 1 \in \operatorname{Spec} D f_a^n(t_1, t_2, s_1 t_3, s_3 t_4) & \end{aligned}$$

It is easy to put the first equation in terms of the map $\vec{x}$:

$$x(s_1 t_1, s_3 t_2, t_3, t_4, a) - (t_1, t_2, s_1 t_3, s_3 t_4, a) = 0.$$

It is also possible to put the second condition in terms of the derivatives of $\vec{x}$ and as a function of $(s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4)$. The rest is also similar to the area preserving case. When $s_1 = s_2 = s_3 = s_4 = 0$ we have a solution of the system of equation given by the existence of the homoclinic tangency. Then, using that the tangency is nondegenerate and unfolds generically, we can apply the implicit function theorem to the system. Hence, we obtain maps $a = a(s_1, s_2, s_3, s_4)$, $t_i = t_i(s_1, s_2, s_3, s_4)$, for $i = 1, \ldots 4$ and such that: $a(0,0,0,0) = a_0$, $t_i(0,0,0,0) = \bar{t}_i$. Using these properties one easily verifies the thesis of the theorem. $\Box$

Acknowledgements. This work has been partially supported by a DGICYT grant no. PB90-0580 (Spain).

References

[1] A. Banyaga, R. de la Llave, C. E. Wayne: Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem. Preprint 1994.

[2] S. E. Newhouse: Generic properties of conservative systems, in *Chaotic behaviour of deterministic systems*, Les Houches 1981 (ed. G. Iooss, R. H. G. Helleman, R. Stora). North Holland (1983).

[3] J. Palis, F. Takens: *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations*. Cambridge studies in advanced mathematics **35**. Cambridge University Press (1993).

[4] J. C. Tatjer, C. Simó: Basins of attraction near homoclinic tangencies. *Ergod. Th. & Dynam. Sys.* (1994), **14**, pp. 351-390.