# TRANSFER ORBITS GUIDED BY THE UNSTABLE/STABLE MANIFOLDS OF THE LAGRANGIAN POINTS

**Annelisie Aiex Corrêa**[1], **Gerard Gómez**[2], **Teresinha J. Stuchi**[3]

[1] DMC/INPE - São José dos Campos, Brazil
[2] MAiA/UB - Barcelona, Spain
[3] IF/UFRJ - Rio de Janeiro, Brazil

e-mail: anne@dem.inpe.br, gerard@maia.ub.es, tstuchi@if.ufrj.br

**Abstract** The unstable and stable manifolds of the Lagrangian point orbits provide a natural mechanism to transfer natural and artificial bodies in the Solar System. In the case of spacecrafts, low energy transfer trajectories can be attained using the complex dynamics described by the unstable/stable manifolds which coalesce in those orbits. However, these manifold tubes do not approach the larger primary, so that is not possible to determine a transfer orbit from the Earth to the Moon vicinity in the Earth-Moon system. This fact can be overcome by decoupling the restricted four body problem into two planar restricted three body systems with a common primary body (Sun-Earth-spacecraft+Earth-Moon-spacecraft). The spacecraft leaves the Earth parking orbit through the stable/unstable manifold structure in the Sun-Earth problem and it is then connected to a transit orbit related to the stable manifold of the Earth-Moon problem. A Poincaré map located on a plane through the Earth is used to find the appropriate connections which depend on the Jacobi's constant of each model.

## 1 Introduction

The restricted three body problem (RTBP) is the simplest chaotic $n$-body problem which fits in first approximation several dynamical system of the Solar System, like the Earth-Moon and the Sun-Earth systems. The linear analysis shows that the collinear equilibrium points of the RTBP, $L_1, L_2$ and $L_3$, are of type saddle $\times$ center $\times$ center, leading the unstable solutions in their vicinity. Although this unstable character, the elliptic directions provide periodic and quasi periodic solutions in their vicinity [1, 2] such as Lyapunov and halo orbits, the Lissajous and other quasi periodic solutions.

In the 1970s, aerospatial engineers began the exploration of these orbits. They were proposed as good places to locate solar observatories due to two main reasons: the point $L_1$ provides uninterrupted access to the solar visual field without occultation by the Earth; and, in these places the solar wind is beyond the influence of the Earth's magnetosphere. The first satellite in a halo orbit was *ISEE-3*, launched in 1978 by NASA. The second mission was the *Soho* telescope projected by ESA-NASA, launched at 1996 for solar observations; *Ace* satellite was launched at 1997 by NASA for solar wind observations. In 2001 two NASA satellites arrived at halo orbits: the *WMap* satellite, to observe cosmic microwave background radiation, and *Genesis*, another solar observatory whose re-entry occurred in 2004.

The unstable character of the collinear Lagrangian points (and their associated orbits) allows the determination of the tangent space to the invariant unstable/stable manifold at the starting point in the periodic orbit [3]. The trajectories on the unstable manifold move away from the vicinity of the center manifold as opposed to the stable manifold, where the trajectories approach the center manifold asymptotically. Since the trajectories of these manifolds follow dynamical natural paths, they are very useful to provide low energy transfer orbits, specially to send a spacecraft to the libration point orbits [4] or, in order to have a transfer between the primaries. The first mission to the libration point orbit which was guided by the invariant manifold tubes was the Genesis mission in 2001-2004.

By decoupling the restricted four body system into two RTBP it is possible to have a natural transfer between the less massive primary of both models, since the manifold tubes do not approach the biggest primaries (in fact, just the manifolds of large amplitude periodic orbits approach the smallest primary body). In our case, we consider the Sun-Earth-spacecraft and Earth-Moon-spacecraft systems. The spacecraft leaves the Earth parking orbit through a stable/unstable manifolds in the Sun-Earth system, makes a swing around $L_1$ (or $L_2$), then it is connected to stable manifold of the Earth-Moon system. The connecting point between the manifolds of these two systems is determined on a appropriate Poincare surface of section, where the tubes determine closed regions.

The main motivation of this kind of transfer procedure has appeared during the Japanese Hiten mission in 1991. The Hiten satellite was supposed to go to the Moon's orbit by a conventional method of transferring but its propellant budget did not permit it. To save this mission a possible solution was to get a low energy transfer with a ballistic capture at the Moon. This solution was accomplish based on the reference [5]. The works [6] and [7] also study this problem. Another application using the separate RTBP has been studied by Gómez et al. [8]. The invariant manifolds were used to construct new spacecraft trajectories to the Jupiter's moons. Also, the manifold structures can explain the phase space conduits transporting material between primary bodies for separate three-body systems.

In this work we present low transfer trajectories between two primary bodies, Earth and Moon, following the unstable/stable invariant structures. The basic ideas are: to decouple the planar restricted four-body problem into two planar RTBP with a common primary (Sun-Earth and Earth-Moon); to determine the unstable/stable invariant manifolds associated to the Lyapunov orbits of $L_1$ and $L_2$ of both systems; to determine the connections between these manifolds on the Poincaré section; and, to study these intersections with respect to the Jacobi's constant of each system.

## 2 Equations of Motion

In the RTBP, the mass of one of the bodies is supposed to be infinitely small when compared to the other two, which move in circular motion around their center of mass. The reference frame is set according to the notation defined in [9]. In the dimensionless coordinate system, the unit of length is the distance between the primaries and the unit of time is chosen in order to have the period of the primaries equal to $2\pi$; consequently, the gravitational constant is set to one. The potential function of the RTBP in this synodical system is given by

$$\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu) \tag{1}$$

where $r_1$ and $r_2$

$$r_1^2 = (x-\mu)^2 + y^2 \qquad \text{and} \qquad r_2^2 = (x-(\mu-1))^2 + y^2$$

are the distances from the primary bodies to the massless particle. The equations of motion are:

$$\ddot{x} - 2\dot{y} = \Omega_x \qquad \ddot{y} + 2\dot{x} = \Omega_y \tag{2}$$

One of the useful characteristic of such formulation is the presence of the first integral known as Jacobi integral and it is related to the potential function by

$$C_J(x, y, \dot{x}, \dot{y}) = -(\dot{x}^2 + \dot{y}^2) + 2\Omega(x, y).$$

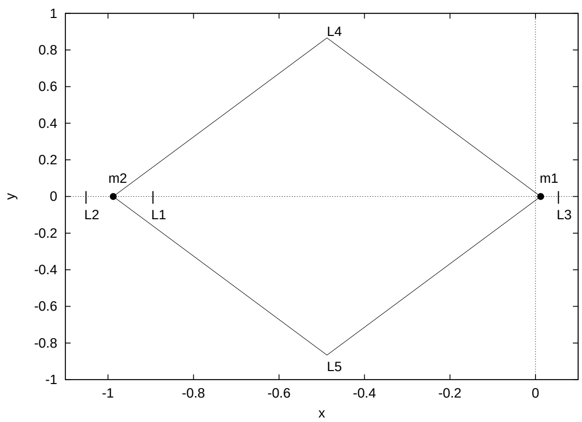


Figure 1: There are three collinear equilibrium points denoted by $L_1$, $L_2$ and $L_3$, and two triangular equilibrium points, $L_4$ and $L_5$.

## 3 Dynamics around Lyapunov Orbits

To analyze the dynamics near a Lyapunov orbit, or the vicinity of any orbit, we must study the variational equations along it. The first order variational equations are

$$\dot{\mathbf{A}} = Df(\mathbf{x}(t))\mathbf{A} \tag{3}$$

where $f(\mathbf{x}(t))$ is the force-field defined by the equations of motion (2) evaluated along the solution $\mathbf{x}(\mathbf{t})$. The solution $\mathbf{A}(t)$ of the variational equations after one period, beginning with $A(0) = I_{4\times 4}$, is known as the monodromy matrix. The linear behavior of the solutions in a vicinity of the periodic orbits is given by the eigenvalues and the corresponding eigenvectors of the monodromy matrix.

In our case, the real eigenvalue pair, with $\lambda_1\lambda_2 = 1$, gives the linear unstable/stable character of the periodic solution and the eigenvector associated to the dominant eigenvalue of this pair gives the expanding direction. At each point of the periodic solution this eigenvector together with the vector tangent to the orbit span a plane tangent to the local unstable manifold. The other two eigenvalues, $\lambda_3 = \bar{\lambda}_4$, are complex conjugated with modulus one. We recall that, in the planar case, the Lyapunov orbits constitute the central manifold.

For the numerical computation of the unstable and stable manifolds of the libration point orbit, it is enough to obtain the dominant eigenvalue and eigenvector of monodromy matrix. The local stable manifold direction is directly determined from the unstable one by the symmetry $(x(t), y(t), \dot{x}(t), \dot{y}(t)) \rightarrow (x(-t), -y(-t), -\dot{x}(-t), \dot{y}(-t))$ of the RTBP.

Given the local approximation, the next step is to procedure the globalization of the unstable manifold. Taking a displacement, $\epsilon$, from a selected point on the Lyapunov orbit $\mathbf{x}(t_i)$, the initial conditions in the linear approximation of the unstable manifold are given by:

$$\mathbf{x}(t_i) \pm \epsilon\hat{\mathbf{e}}(t_i)$$

where $\hat{\mathbf{e}}(t_i)$ is the normalized unstable direction at $\mathbf{x}(t_i)$. The points $\mathbf{x}(t_i)$ should be equally spaced on the periodic orbit to guarantee no gaps in the Poincaré section. Since we are dealing with unstable periodic orbits, each point is locally a saddle point having two unstable branches, which depend on the $\epsilon$ sign. Note that, for the globalization of the stable manifold the procedure is the same, but one should integrated backwards.

## 4 Transfer in/out the Hyperbolic Invariant Manifolds

As seen before, the linear stability of the libration point orbits allows to design paths on the stable and unstable invariant manifold tubes. To easily understand the flow inside and outside these tubes, consider a linear change of variables which cast the second order Hamiltonian in the form

$$H_2 = \lambda q_1 p_1 + \frac{\omega}{2}(q_2^2 + p_2^2)$$

where $\lambda$ and $\omega$ are positive real values; $q_i$ and $p_i$ are the new canonic coordinates (see [2] for details). Considering this linear equations of motion we can easily uncouple the saddle and the center directions. The Cartesian product of the asymptotic hyperbolic direction with the periodic orbit form local cylinders lying in the phase space. However, these cylinders are distorted by the non linear terms of the Hamiltonian field resulting the stable and unstable manifolds which can cross each other at homoclinic or heteroclinic intersections.

Since the energy surface is 3-dimensional in the planar RTBP, the 2-dimensional manifold tubes act as separatrices of the phase space (see [5]-[8]). Therefore, the trajectories can be classified in two types: the transit ones, which move inside the two-dimensional manifold tubes, and the non transit trajectories, which are those outside the tubes. Moreover, the transit trajectories travel between the exterior and interior of the Hill's regions, and the non transit trajectories remain just in one side of the Hill's region (Figures 2 and 3 illustrate both trajectories).

The initial conditions of a non transit orbit are obtained through the Poincaré map ($x = -1+\mu_{SE}$) taking a point outside the unstable manifold cut and correcting the component $\dot{x} = \dot{x}(y, \dot{y}, C_J)$ to take it at same energy level of the manifold. A backwards integration maps this point to the vicinity of the corresponding stable manifold which is close to the Earth, where the launching point is determined (see Figure 2). Note that a trajectory started in this launching point near the Earth returns to its neighborhood after a loop around $L_2$.

The transit trajectory is obtained in a similar way but with initial conditions inside the manifold cut on the Poincaré section. This procedure is shown in Figure 3 (left), and the corresponding trajectory is presented in Figure 3 (right) for the Earth-Moon system.

## 5 Superposition of Two RTBP

Recall that the mean idea is to decompose the restricted four body problem, Sun-Earth-Moon-spacecraft into two coupled RTBP with a common primary body, Sun-Earth-spacecraft and Earth-Moon-spacecraft. The trajectory design is divided into two paths: firstly, the spacecraft leaves the Earth vicinity through a non transit trajectory in the Sun-Earth system, then it is connected to the appropriate transit trajectory in the Earth-Moon system. In the transfer first part, the transit trajectory could be associated to the $L_1$ or $L_2$ hyperbolic invariant manifold, but in the second one, it must belong the $L_2$ hyperbolic invariant manifold,

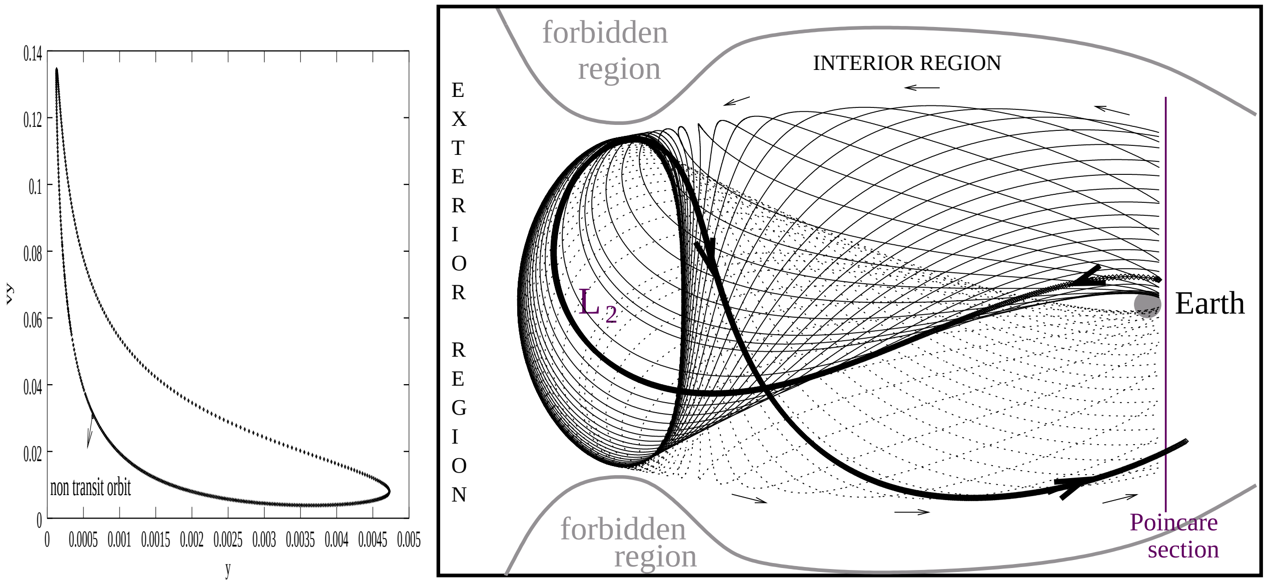


Figure 2: Left: the Poincaré section of the unstable manifold and the initial conditions of a non transit orbit associated to $L_2$-manifold tubes of the Sun-Earth system. Right: the dark trajectory is the non transit trajectory. This trajectory allows the first path of the transfer, in which the spacecraft leaves the parking orbit guided by the dynamics of $W_{SE}^{s,u}$.

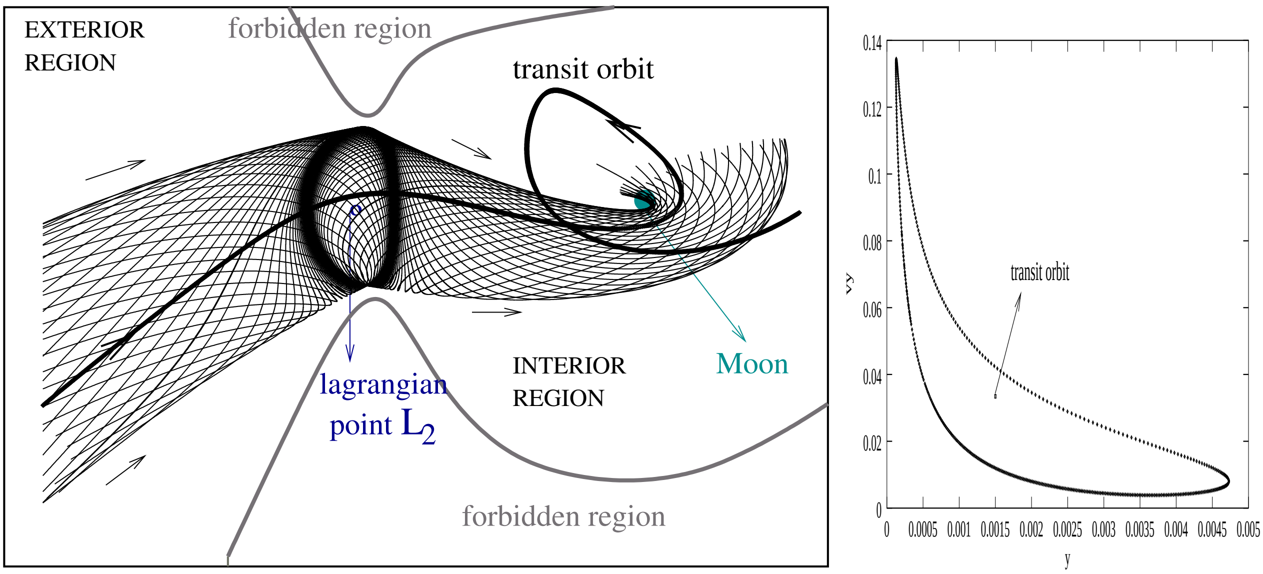


Figure 3: Right: the Poincaré section of the stable manifold and the initial conditions of a transit orbit associated to $L_2$-manifold tubes of the Earth-Moon system. Left: the related transit trajectory.

because the lobe of the Hill's regions opened first at $L_1$ providing communication only between the inner regions.

As already mentioned, the connection point is determined on the Poincaré map through a plane passing on the Earth. In the Sun-Earth system it is located at $x = -1 + \mu_{SE}$ and, in the Earth-Moon system, it is located at $x = \mu_{EM}$ ($\mu_{SE} = 3.040357143 \times 10^{-6}$ and $\mu_{EM} = 0.0121505816$). To adjust these two surfaces we must change the coordinates adequately, writing the coordinates of the Earth-Moon system in terms of the Sun-Earth system.

In order to examine possible connections between the $W^u_{SE}|_{L_2}$ and $W^s_{EM}|_{L_2}$, we vary the Jacobi constant of each model. Firstly, we define a sphere of radius $R = 0.00020$ (Sun-Earth unit) and $R = 0.013028$ (Earth-Moon unit) around the Earth and Moon, respectively. We discard the hyperbolic invariant manifolds which pass outside these spheres, because the associated Lyapunov orbits do not allow strategic parking orbits. The Lyapunov family of the Sun-Earth system which allows the desired unstable invariant manifold cut on the Poincaré section, belonging to the Jacobi's constant range of $[3.00079083, 3.0005689]$. For the Earth-Moon system this range is $[3.14962509, 3.0654849]$. The corresponding Lyapunov orbits are shown in Figure 4. The intersections of the associated hyperbolic invariant manifolds on the Poincaré section are shown in Figure 5.

To propagate the non transit and the transit trajectories, we have chosen an arbitrary point $P_1$ in the intersection zone (Figure 6, top). From this Poincaré map, the initial conditions of $P_1$ must be backward integrated to generate the non transit and the transit one, which follows the dynamics inside the stable manifold and should be forward integrated. At this Poincaré section, the non transit and the transit trajectories have not exactly the same coordinates, the $\dot{x}$ component is not equal due to the decoupling of the restricted four body problem. This can be overcome by adjusting this velocity, requiring a small velocity increment.

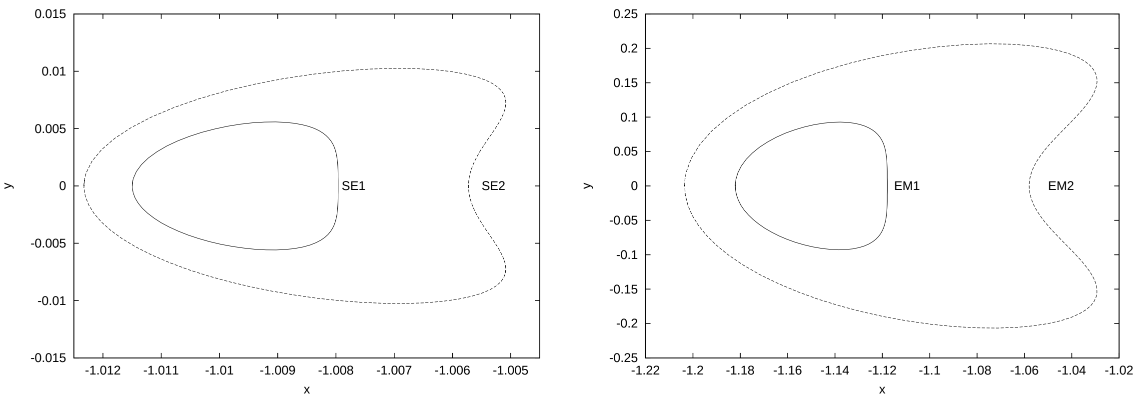


Figure 4: Limiting Lyapunov orbits around $L_2$ in the Sun-Earth (SE) and Earth-Moon (EM) systems, respectively. Their hyperbolic invariant manifolds give the desired zone of intersection between the manifolds tubes in both RTBP models. The index $1, 2$ are related to the Jacobi's constant which defines the intersection zone (see Table 1).

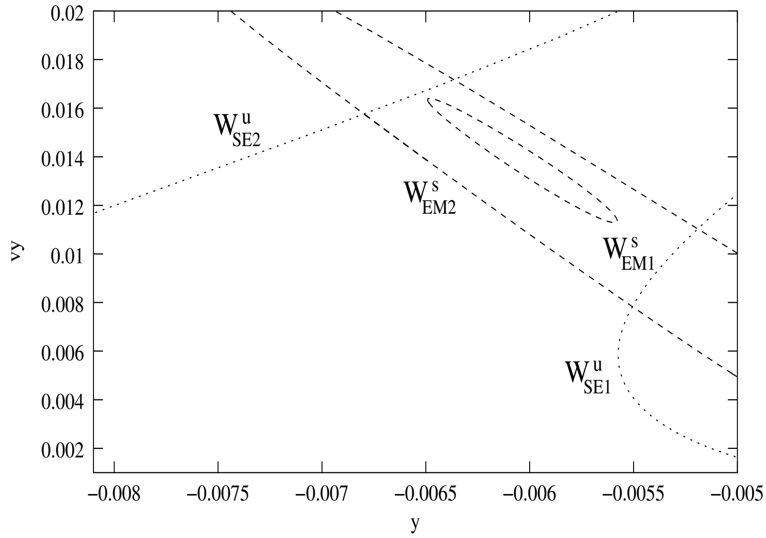


Figure 5: Poincaré section of the invariant manifold tubes of the limiting Lyapunov orbits. The bored losangle is the intersection zone. $W^u_{SE}$: unstable invariant manifold of the $L_2$-center manifold in the Sun-Earth system; and, $W^s_{EM}$: stable invariant manifold of the $L_2$-center manifold in the Earth-Moon system.

Table 1: Initial conditions, Jacobi's constant, stability parameter and period of the $L_2$-Lyapunov orbits in both system.

| | $x_0$ | $\dot{y}_0$ | $C_J$ | $k_1$ | $k_2$ | T/2 |
|---|---|---|---|---|---|---|
| SE1 | -1.01149819 | 0.01093317 | 3.00079083 | 1560.83607 | 2.02573493 | 1.5558992 |
| SE2 | -1.020392633 | 0.35059982 | 3.0005689 | 886.953969 | 2.19676591 | 1.67918158 |
| EM1 | -1.18212003 | 0.16488212 | 3.14962509 | 1184.34113 | 2.00859114 | 1.69580095 |
| EM2 | -1.20351928 | 0.3476276 | 3.0654849 | 488.94977 | 2.28655205 | 1.87911492 |

After this procedure, the transit trajectory is integrated in the Sun-Earth system giving the desired transfer trajectory to the Moon (see Figure 6, bottom).

# 6 Comments

The transfer of a spacecraft guided by the stable and unstable invariant manifolds always provides a low transfer cost. In our case the coupled RTBP allowed a low cost transfer path to the Moon, requiring an adjust of velocity of 0.0652245 canonical units (Sun-Earth), which is much less than a conventional maneuver.

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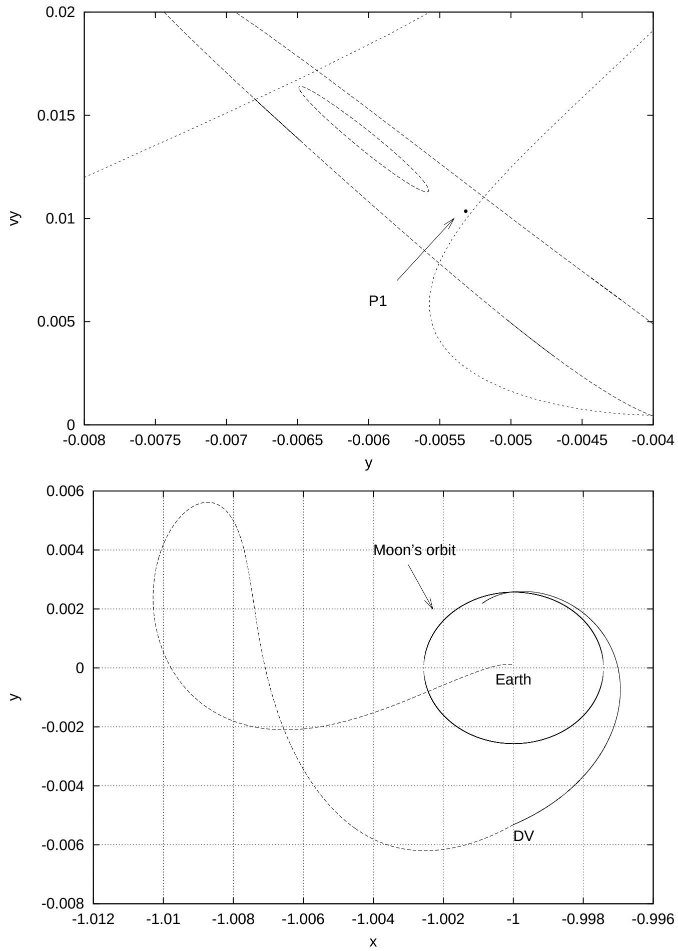


Figure 6: The top figure shows the intersection zone. $P1$ represents the initial condition of the transit The figure bellow is the propagate trajectory on the $(xy)$ plane in the Sun-Earth synodical reference frame.

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