

Evidence and conditions for non-integrability in the Friedmann-Robertson-Walker Hamiltonian

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Abstract

This is an example of application of Ziglin-Morales-Ramis algebraic studies in Hamiltonian integrability, more specifically the result by Morales, Ramis and Simó on higher-order variational equations, to the well-known Friedmann-Robertson-Walker cosmological model. A previous paper by the author formalises said variational systems in such a way allowing the simple expression of notable elements of the differential Galois group needed to study integrability. Using this formalisation and an alternative method already used by other authors, we find sufficient conditions whose fulfilment would entail very simple proofs of non-integrability – both for the complete Hamiltonian, a goal already achieved by other means by Coelho et al, and for a special open case attracting recent attention.

1 Introduction

1.1 The Friedmann-Robertson-Walker Hamiltonian

The *Friedmann-Robertson-Walker* (FRW) Hamiltonian in its conformal metric form [7] is

$$H(\mathbf{Q}, \mathbf{P}) := \frac{1}{2} \left[- (P_1^2 + kQ_1^2) + (P_2^2 + kQ_2^2) + m^2 Q_1^2 Q_2^2 + \frac{\lambda}{2} Q_2^4 + \frac{\Lambda}{2} Q_1^4 \right],$$

where m is the mass, Q_1 is the scale factor, $k = 0, \pm 1$ is the curvature, Λ is the cosmological constant, λ is the self-coupling of Q_2 and \mathbf{P} are the conjugate momenta of \mathbf{Q} . Canonical change $Q_1 = -iq_1$, $P_1 = ip_1$, $Q_2 = q_2$ and $P_2 = p_2$, as mentioned in [6], transforms this into

$$H(\mathbf{q}, \mathbf{p}) := \frac{1}{2} (p_1^2 + p_2^2) + \frac{k}{2} (q_1^2 + q_2^2) + \frac{\Lambda q_1^4}{4} - \frac{1}{2} m q_1^2 q_2^2 + \frac{\lambda q_2^4}{4}. \quad (1)$$

So far, the state of the art on the subject is as follows:

Theorem 1.1.1 ([6]). *Consider special values*

$$\{\Lambda, \lambda\} \cap \{\mu(n) : n \in \mathbb{Z}\} \neq \emptyset, \quad \mu(p) := -\frac{2m^2}{(p+1)(p+2)}. \quad (2)$$

Hamiltonian (1) for $k \neq 0$ is integrable for $\Lambda = \lambda$, $n = 0, 1$ above. It is non-integrable for any other values of Λ, λ save, perhaps, cases $n > 10$ for either parameter in (2).

Coelho et al. later eliminated the remaining cases (2) by means of the Kovacic algorithm:

Theorem 1.1.2 ([7, Th. 5]). *Hamiltonian (1) for $k \neq 0$ is non-integrable save for $\Lambda = \lambda$ and $n = 0, 1$ in the aforementioned cases (2): $\Lambda = \lambda = -m^2/3$ or $\Lambda = \lambda = -m^2$.*

The aforementioned article tackled case $k = 0$ as well, capitalising on the fact that (1) becomes classical with a homogeneous potential and conjecturing possible integrability for a few special cases using an 18-item table by Morales and Ramis in [16].

Section 2 of this paper offers a sufficient condition which could in turn be used to obtain a much shorter proof of Theorem 1.1.2 using higher-order variational equations (a tool already used in [6]) along solution (5). In Section 3 we will then perform the same task on the conjectured cases for integrability for $k = 0$ in [7] to obtain an analogous sufficient condition for non-integrability. These new results are not proofs of non-integrability in and of themselves, but said proofs are the logical next step and will be dealt with in further theoretical studies.

The techniques used not only complement those used in [7], but add a degree of specificity to forthcoming proofs of non-integrability; they will also be used on further studies from an alternative approach by the author, Aparicio et al in [2] on formal expansions of first integrals.

1.2 The algebraic study of Hamiltonian integrability

1.2.1 Basics

Let $\psi(t, \cdot)$ be the flow and $\phi(t)$ be a particular solution of a given autonomous system

$$\dot{z} = X(z), \quad X : \mathbb{C}^m \rightarrow \mathbb{C}^m, \quad (3)$$

respectively. The **variational system** of (3) along ϕ has $\frac{\partial}{\partial z}\psi(t, \phi)$ as a fundamental matrix:

$$\dot{Y} = A_1 Y, \quad A_1(t) := \left. \frac{\partial X}{\partial z} \right|_{z=\phi(t)} \in \text{Mat}_n(K), \quad (\text{VE}_\phi)$$

$K = \mathbb{C}(\phi)$ being the smallest differential field containing entries of solution $\phi(t)$. In general, $\frac{\partial^k}{\partial z^k}\psi(t, \phi)$ are multilinear k -forms appearing in the Taylor expansion of the flow along ϕ :

$$\psi(t, z) = \psi(t, \phi) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \psi(t, \phi)}{\partial z^k} \{z - \phi\}^k;$$

$\frac{\partial^k}{\partial z^k}\psi(t, \phi)$ also satisfy an echeloned set of systems, depending on the previous $k - 1$ partial derivatives and usually called **order- k variational equations** VE_ϕ^k . Thus we have, (3) given, a *linear* system $\text{VE}_\phi =: \text{VE}_\phi^1 =: \text{LVE}_\phi^1$ and a family of *non-linear* systems $\{\text{VE}_\phi^k\}_{k \geq 2}$.

In [19] the author presented an explicit *linearised* version LVE_ϕ^k , $k \geq 1$, by means of symmetric products \odot of finite and infinite matrices based on already-existing definitions by Bekbaev, e.g. [4]. This was done in preparation for the Morales-Ramis-Simó (MRS) theoretical framework appearing in Section 1.2.2 below, but has other consequences as well. More specifically, our outcomes in [19] have two applications for system (3), *Hamiltonian or not*:

- full structure of VE_ϕ^k and LVE_ϕ^k , i.e. *recovering the flow*, which underlies the MRS theoretical corpus in practicality;
- a byproduct is the full structure of dual systems $(\text{LVE}_\phi^k)^\star$, i.e. *recovering formal first integrals of (3)* in ways which simplified earlier results in [2] significantly.

Numerical and symbolic computations in the present paper are based on the first of these applications. Applications of the techniques in the second item to the FRW Hamiltonian will also be the subject of imminent further work. The result in [19] that attracts our immediate attention is

a succinct summary. The heuristics of the theory are couched on the principle that an integrable system, Hamiltonian or otherwise, must yield first-order variational systems (VE_ϕ) which are integrable (in the Galoisian sense) along any non-constant solution ϕ . For Hamiltonian systems, this principle has the following incarnation:

Theorem 1.2.2. (MORALES-RUIZ & RAMIS, 2001). *If (3) is Hamiltonian and meromorphically integrable in a neighbourhood of non-constant solution ϕ , then the Zariski identity component $\text{Gal}(\text{VE}_\phi^1)^0$ is commutative.*

A further result coauthored by Simó extended the result to higher-order variationals:

Theorem 1.2.3. (MORALES-RUIZ, RAMIS & SIMÓ, 2005). *Along the previous lines, let $G_k := \text{Gal}(\text{VE}_\phi^k)$, $k \geq 1$, and $\widehat{G} := \varprojlim G_k$ their inverse limit. Then, the identity components of the Galois groups G_k and \widehat{G} are abelian.*

Their proof used jet analysis on VE_ϕ^k and the fact that all linearisations thereof by means of additional variables are equivalent (hence the unambiguous use of $\text{Gal}(\text{VE}_\phi^k)$), but they avoided the need to systematise one such linearisation. Our LVE_ϕ^k summarised in 1.2.1 does the latter step and is a step forward towards a constructive version of Theorem 1.2.3 and its applications.

2 Case $k \neq 0$

Let us now address the FRW Hamiltonian with the above theoretical tools. As said in the introduction, we first exemplify our procedure on a possible simplification of an already-existing proof. $k = 1$ will be not only our paradigm but also the only case worth studying, since calculations are similar for $k = -1$. We invite the reader to check this.

Assume $k = 1$, therefore, and consider the family of invariant-plane particular solutions

$$q_2 \equiv p_2 \equiv 0, \quad q_1(t) = \frac{\text{i}\sqrt{1 + \sqrt{1 + 2\Lambda C_1}} \text{sn}\left(\frac{\sqrt{1 - \sqrt{1 + 2\Lambda C_1}}(t + C_2)}{\sqrt{2}}, \frac{1 + \sqrt{1 + 2\Lambda C_1}}{1 - \sqrt{1 + 2\Lambda C_1}}\right)}{\sqrt{\Lambda}}, \quad p_1 \equiv \dot{q}_1,$$

where sn is the *Jacobi elliptic sine function* [1, Ch. 16]. Assigning special values to C_1 and C_2 we obtain bifurcations into simpler functions, for instance for $C_1 = -\frac{1}{2\lambda}$

$$\phi(t) = \left(-\frac{\text{i}}{\sqrt{\Lambda}} \tanh \frac{t}{\sqrt{2}}, 0, -\frac{\text{i}}{\sqrt{2\Lambda}} \text{sech}^2 \frac{t}{\sqrt{2}}, 0\right), \quad (5)$$

having period $\text{i}\sqrt{2}\pi$ and poles $\frac{\text{i}\pi(2j+1)}{\sqrt{2}}$, $j \in \mathbb{Z}$. Work in Section 2 will be based on solution (5).

2.1 Numerical evidence of non-integrability

Let ϕ be the solution in (5), assume $m = 1$ and assume all parameters equal the same exceptional value in (2), $\Lambda = \lambda = -\frac{2m^2}{(p+1)(p+2)}$, for the sake of concreteness. Consider any two paths γ_1 and γ_2 in the complex plane \mathbb{C} containing singularities $t^* = \text{i}\pi/\sqrt{2}$ and $-t^*$ respectively. For $i = 1, 2$ analytic continuation of any solution Φ_k of LVE_ϕ^k along $t \in \gamma_i$ yields monodromy matrix M_{k,γ_i} . Define commutators $C_k = M_{k,\gamma_1} M_{k,\gamma_2} - M_{k,\gamma_2} M_{k,\gamma_1}$. Our simulation relates p (seen as a general real number) to both commutators C_k and the deviation of M_{k,γ_i} from the identity matrix $\text{Id}_{d_{n,k}}$. The reason for the latter is that trivial first-order monodromies M_{1,γ_i} automatically belong to $\text{Gal}(\text{VE}_\phi^1)^\circ$ and M_{k,γ_i} will also belong, therefore, to $\text{Gal}(\text{LVE}_\phi^k)^\circ$ since quadratures only affect the identity component of the Galois group and solutions to higher-order variationals, in theory, only require computing integrals.

First conclusions based on preliminary numerical experiments on random paths, e.g. Figure 1 for squares $\gamma_{1,2}$ having vertices $\pm 4\text{i}$, $\pm 2 \pm 2\text{i}$, $\pm 2 \mp 2\text{i}$, are:

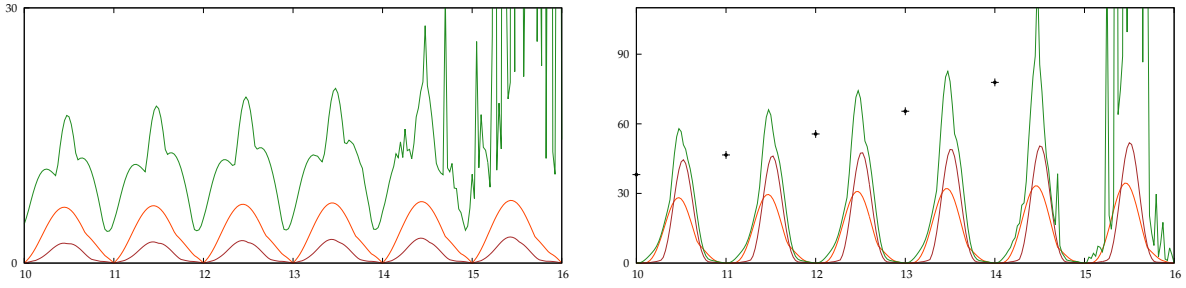


Figure 1: Detachment from $\text{Id}_{d_{n,k}}$ and commutators for monodromies LVE_{ϕ}^k along $\gamma_1, \gamma_2, p \in [10, 16]$

- non-trivial monodromies first appear for $k = 3$, hence both monodromies $M_{k,\gamma_1}, M_{k,\gamma_2}$ for all orders belong to the connected component of the corresponding Galois groups;
- for integer values of p , obstructions to integrability first arise at order $k = 5$; $\|C_4\|_{\infty}$ is almost uniformly bounded by 10^{-10} which numerically counts as vanishing at that order;
- the growth of the entries in the solution matrix to VE_{ϕ}^1 , hence in blocks arising from quadrature or symmetrical products, calls for an adjustment of the paths in order to curb error propagation for increasing values of p , as shown for values $p > 14$ in both figures.

A first step to check the veracity of each of the above is choosing paths for which the sub- ∞ norm of the solution to VE_{ϕ}^1 is close to minimal. Such are, for instance, γ_1 given by hexagon with vertices $\{0, \frac{6}{5} + \frac{6}{5}i, \frac{6}{5} + (\sqrt{2}\pi - \frac{6}{5})i, \sqrt{2}\pi i, -\frac{6}{5} + (\sqrt{2}\pi - \frac{6}{5})i, -\frac{6}{5} + \frac{6}{5}i\}$ and $\gamma_2 = -\gamma_1$. Our simulations then yield the results shown in Figure 2.

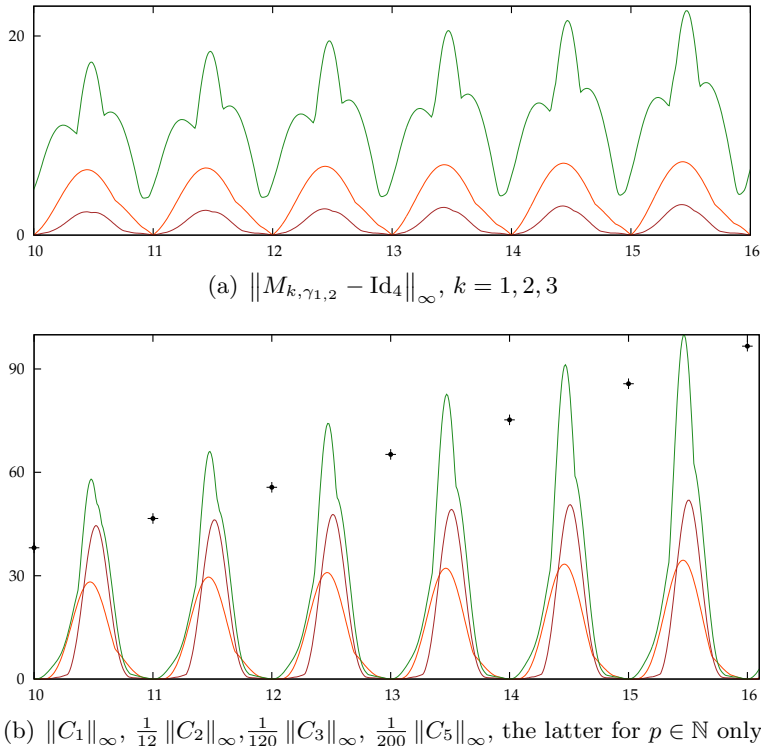


Figure 2: Detachment from $\text{Id}_{d_{n,k}}$ and commutators for monodromies LVE_{ϕ}^k along new paths

All of the above seems to point out that obstructions to integrability for H_{FRW} arise in LVE_{ϕ}^5 . In Theorem 1.1.1, Weil and Boucher used the full power of non-linearised third-order

variational equations along an invariant-plane solution complementing (5) to narrow the proof into exceptional cases (2). Our simulations strongly suggest order-4 variational equations would have not changed this scenario, hence our attention will focus on order 5 directly.

2.1.1 Using numerics to alleviate workload in symbolic calculations

In the simulations leading to Figures 1 and 2, recursive jet computation along two separate paths J_{k,γ_1} , J_{k,γ_2} as in [10, 11, 12], followed by linearisation into M_{k,γ_1} , M_{k,γ_2} as in [19] and computation of commutators $C_k = M_{k,\gamma_1}M_{k,\gamma_2} - M_{k,\gamma_2}M_{k,\gamma_1}$, was the preferred course of action for computational reasons. Indeed, the other practical method would be computing a single jet $J_{k,\gamma}$ for $\gamma = \gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$ – that is, using only the methods described in [10, 11, 12], and the fact

$$M_{k,\gamma} = M_{k,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1} = M_{k,\gamma_2}^{-1}M_{k,\gamma_1}^{-1}M_{k,\gamma_2}M_{k,\gamma_1}, \quad \text{for every } k \geq 1, \quad (6)$$

to check whether jet $J_{k,\gamma}$, i.e. the lower four rows of $M_{k,\gamma}$, equals $(0_{4 \times d_{4,k}}, \dots, 0_{4 \times d_{4,2}}, \text{Id}_4)$. Since integrating LVE_ϕ^5 along four consecutive complex paths and then linearising is computationally more expensive than doing so in only two paths, linearising and then multiplying and subtracting 125×125 matrices, this was definitely discarded as a choice for Section 2.1.

However, a rigorous proof calls for as few symbolic computations as possible, which does call for monodromies along $\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$. Furthermore, the information gleaned from numerical computations in Section 2.1 will be useful to us in the following way.

If instead of computing commutators for the numerical order-five monodromies M_{5,γ_1} , M_{5,γ_2} leading to Figures 2(a) and 2(b), we perform the more costly operation of computing the monodromy for their path commutator as in (6), subtract Id_{125} and cap all numbers below 10^{-9} to zero, we have the following numbers left in its four lower rows $K := J_{5,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1} \in \text{Mat}_{4 \times 125}$:

$$K_{2,38}, \quad K_{2,45}, \quad K_{2,56}, \quad K_{4,36}, \quad K_{4,41} = -K_{2,38}, \quad K_{4,50} = -K_{2,45}. \quad (7)$$

all of them unsurprisingly among the first 56 columns (i.e. no obstructions for $k \geq 4$) and all pure imaginary numbers. $K_{4,36}$ is consistently the coefficient with the largest modulus.

All we need for symbolic computations in 2.2.2, therefore, is this information on $K_{4,36}$. The non-vanishing of this term will be our rigorous sufficient condition for non-integrability.

2.2 Condition for non-integrability

2.2.1 First-order variational

The above numerical evidence implied the triviality of first-order monodromy matrices for $\Lambda \in \mu(\mathbb{Z})$. Let us first prove this rigorously. Variational equations (VE_ϕ) along ϕ as in (5) split into

$$\ddot{\xi}_1 = \left(-1 + 3 \tanh^2 \frac{t}{\sqrt{2}}\right) \xi_1, \quad \ddot{\xi}_2 = - \left(1 + \frac{m^2 \tanh^2 \frac{t}{\sqrt{2}}}{\Lambda}\right) \xi_2. \quad (8)$$

Using algebrisation [3], transformations $t = \sqrt{2} \operatorname{arctanh} x$ and $t = i\sqrt{2} \operatorname{arctan} \frac{\sqrt{\Lambda+x}}{m}$ on (8) yield the following principal fundamental matrix for (VE_ϕ) :

$$\Phi(t) := \begin{pmatrix} f_1 & 0 & f_2 & 0 \\ 0 & g_1 & 0 & g_2 \\ \dot{f}_1 & 0 & \dot{f}_2 & 0 \\ 0 & \dot{g}_1 & 0 & \dot{g}_2 \end{pmatrix},$$

whose first and third columns are given by functions belonging to base field $K := \mathbb{C}\left(t, \tanh \frac{t}{\sqrt{2}}\right)$,

$$f_1(t) := \cosh^{-2} \frac{t}{\sqrt{2}}, \quad f_2(t) := \frac{1}{8} \left(3t \cosh^{-2} \frac{t}{\sqrt{2}} + \sqrt{2} \left(\sinh \sqrt{2}t + 3 \tanh \frac{t}{\sqrt{2}} \right) \right), \quad (9)$$

whereas the other two require a non-trivial differential extension. $\Lambda = -\frac{2m^2}{(n+1)(n+2)}$ implies g_1, g_2 can be written in terms of *Legendre associated functions* $P_{n+1}^{\sqrt{n}\sqrt{3+n}}(z), Q_{n+1}^{\sqrt{n}\sqrt{3+n}}(z)$, [1, Ch. 8], with $z = \tanh \frac{t}{\sqrt{2}}$; we leave the details to the reader, simply stating that condition $n \in \mathbb{N}_{n \geq 2}$ renders them expressible in terms of polynomials of degree $n+1$ and irrational powers of $(1 \pm z)$:

$$g_i(t) = G_{n,i} \left(\tanh \frac{t}{\sqrt{2}} \right) \cosh^{-\sqrt{n}\sqrt{3+n}} \frac{t}{\sqrt{2}}, \quad i = 1, 2, \quad (10)$$

where

$$G_{n,i}(z) := H_{n,1}(-z) + H_{n,1}(z), \quad H_{n,i}(z) := (1+z)^{-\sqrt{n}\sqrt{3+n}} \sum_{j=0}^{n+1} a_{i,j} z^j, \quad i = 1, 2. \quad (11)$$

Assume, therefore, $z = \tanh \frac{t}{\sqrt{2}}$ and t transits along a path γ_{\pm} containing either singularity $\pm t^* = \pm \frac{i\pi}{\sqrt{2}}$. $\sum_{j=0}^{n+1} a_{i,j} z^j$ in (11) are easily checked to be entire functions of t . Hence the only possible source of branching in (11), i.e. non-trivial monodromy for (\mathbf{VE}_{ϕ}) , could be:

- $\left(1 \pm \tanh \frac{t}{\sqrt{2}}\right)^{-\sqrt{n}\sqrt{3+n}}$ when t crosses lines $L_+ := \left\{ \operatorname{Im} t = \frac{\pi i}{\sqrt{2}} \right\}, L_- := \left\{ \operatorname{Im} t = -\frac{\pi i}{\sqrt{2}} \right\}$,
- and term $\operatorname{sech}^{\sqrt{n}\sqrt{3+n}} \frac{t}{\sqrt{2}}$ whenever t crosses the imaginary axis outside of 0.

Therefore, given any path γ_{\pm} encircling $\pm t^*$ three notable points prevail: intersections t_1^{\pm} and t_3^{\pm} with L_{\pm} and, right between them, intersection t_2^{\pm} with the positive (resp. negative) imaginary line. Let us choose γ_+ for the sake of simplicity. With regards to the determination of \ln , the effect of t_1^+ and t_3^+ on $\left(1 + \tanh \frac{t}{\sqrt{2}}\right)^{-\sqrt{n}\sqrt{3+n}}$ and $\left(1 - \tanh \frac{t}{\sqrt{2}}\right)^{-\sqrt{n}\sqrt{3+n}}$ respectively, is the opposite of that of t_2 on $\operatorname{sech}^{\sqrt{n}\sqrt{3+n}} \frac{t}{\sqrt{2}}$. Indeed, all three points, t_1^+, t_3^+ and t_2^+ , force the addition of $2\pi i$ to $\ln\left(1 + \coth \frac{t}{\sqrt{2}}\right), \ln\left(1 - \coth \frac{t}{\sqrt{2}}\right)$ and $\ln \operatorname{sech} \frac{t}{\sqrt{2}}$, respectively, but the former two logarithms accompany a negative power $-\sqrt{n}\sqrt{3+n}$, whereas the latter one is linked to positive power $\sqrt{n}\sqrt{3+n}$. Hence, $e^{-2\pi i \sqrt{n}\sqrt{3+n}}$ (from $(1 \pm z)^{-\sqrt{n}\sqrt{3+n}}$) and $e^{2\pi i \sqrt{n}\sqrt{3+n}}$ (from common factor $\operatorname{sech}^{\sqrt{n}\sqrt{3+n}} \frac{t}{\sqrt{2}}$) cancel out in expressions (10) and (11) after point t_3^+ , and functions $g_{n,1}$ and $g_{n,2}$ return to the values at 0 of their original branches.

Hence the monodromy of (\mathbf{VE}_{ϕ}) along any path based at $t = 0$ and encircling $\pm t^*$ is equal to the 4×4 identity matrix, as predicted from numerical evidence in Figure 2(a). This ensures the belonging of higher-order monodromies to the respective identity components of the Galois groups containing them, as said previously.

2.2.2 Higher-order variationals

Using Subsection 2.1.1, and with the six entries (7) of the lower row of $M_{5, \gamma_+^{-1} \gamma_+^{-1} \gamma_-^{-1} \gamma_-^{-1}} - \operatorname{Id}_{125}$ in mind, let us choose $K_{4,36}$, which not only has the simplest symbolic expression in terms of lower-order quadratures (a distinction shared by $K_{2,56}$), but also yields the largest modulus in numerical computations as stated above.

Using $f_1 \dot{f}_2 - f_2 \dot{f}_1 = g_1 \dot{g}_2 - g_2 \dot{g}_1 = 1$, we have $K_{4,36} = \int_{\gamma_-^{-1} \gamma_+^{-1} \gamma_- \gamma_+} A(t) dt$, where

$$A(t) := \frac{20g_1 \cdot \left(-3\Lambda g_1 (2m^6 G_{1,3}^2 + \lambda G_{2,15} g_1) + 2m^4 (G_{1,3} G_{2,15} + 2G_{3,11} g_1) \tanh \frac{t}{\sqrt{2}} \right)}{\Lambda^2},$$

defined in terms of indefinite quadratures and combinations thereof:

$$G_{3,11} = f_2 F_{3,21} - f_1 F_{3,22}, \quad G_{2,15} = 6F_{2,29} g_1 + \Lambda F_{2,30} g_2, \quad G_{1,3} = f_1 F_{1,5} - F_{1,4} f_2,$$

where quadratures given by LVE_ϕ^4 are

$$F_{3,21}(t) = \int f_1(\tau) \left(3\Lambda m^2 G_{1,3}(\tau) g_1(\tau)^2 - (9\Lambda m^2 G_{1,3}(\tau)^2 + G_{2,15}(\tau) g_1(\tau)) \tanh \frac{\tau}{\sqrt{2}} \right) d\tau,$$

$$F_{3,22}(t) = \int f_2(\tau) \left(3\Lambda m^2 G_{1,3}(\tau) g_1(\tau)^2 - (9\Lambda m^2 G_{1,3}(\tau)^2 + G_{2,15}(\tau) g_1(\tau)) \tanh \frac{\tau}{\sqrt{2}} \right) d\tau,$$

quadratures arising from LVE_ϕ^3 are

$$F_{2,29}(t) = \int g_2 \left(\lambda \Lambda g_1^3 - 2m^4 G_{1,3} g_1 \tanh \frac{\tau}{\sqrt{2}} \right) d\tau, \quad F_{2,30}(t) = 6 \int \left(\frac{2m^4 G_{1,3} g_1^2 \tanh \frac{\tau}{\sqrt{2}}}{\Lambda} - \lambda g_1^4 \right) d\tau,$$

and those arising from VE_ϕ^1 are

$$F_{1,4}(t) = \int f_1(\tau) g_1^2(\tau) \tanh \frac{\tau}{\sqrt{2}} d\tau, \quad F_{1,5}(t) = \int f_2(\tau) g_1^2(\tau) \tanh \frac{\tau}{\sqrt{2}} d\tau.$$

We thereby obtain our sufficient condition for non-integrability:

Proposition 2.2.1. *For any value of (Λ, λ) for which $K_{4,36} = \int_{\gamma_-^{-1} \gamma_+^{-1} \gamma_- \gamma_+} A(t) dt \neq 0$, the FRW Hamiltonian is not integrable. \square*

The uniformity of numerical evidence for $k = 5$ (as opposed to that in 3.3 later on) and what we already know from Theorem 1.1.2 ostensibly validate the following:

Conjecture 2.2.2. *$K_{4,36} \neq 0$ for every value of (Λ, λ) except for $n = 0, 1$ in (2). Hence, the order-five variational equations yield the first obstruction to integrability in H .*

The above is but a hint at a simpler, yet somehow more specific proof of an already-known result. Let us now use the same procedure on an open problem.

3 Case $k = 0$: homogeneous potentials and a new result

3.1 Homogeneous potentials

Define $V_4 := \Lambda q_1^4/4 - m^2 q_1^2 q_2^2/2 + \lambda q_2^4/2$. Proving Hamiltonian $H_h := \frac{1}{2} p^2 + V_4(\mathbf{q})$ meromorphically non-integrable save for a few exceptional cases entails the following:

- in virtue of a result by Mondéjar et al. on non-homogeneous polynomial potentials ([13], see also [9, Th 1.1]) the integrability of H in (3) implies that of H_h . We would therefore obtain yet another proof of [7, Th. 5].
- H_h also corresponds to case $k = 0$ in the original Hamiltonian (3). A non-integrability result will show light on the (non)-integrability conjectured in [7, §6].

There are nine solutions to equation $V_4'(\mathbf{c}) = \mathbf{c}$, customarily called *Darboux points* [9]:

$$(c_1, c_2) \in \left\{ (0, 0), \quad \left(\pm\Lambda^{-\frac{1}{2}}, 0\right), \quad \left(\pm\sqrt{\frac{\lambda + m^2}{\lambda\Lambda - m^4}}, \pm\sqrt{\frac{\Lambda + m^2}{\lambda\Lambda - m^4}}\right), \quad \left(\pm\lambda^{-\frac{1}{2}}, 0\right) \right\}.$$

The non-trivial eigenvalues $\alpha_2 \neq 3$ of $V_4''(c_1, c_2)$ for each Darboux point are summarised below:

$$\alpha_{2,1} = 0, \quad \alpha_{2,2} = -\frac{m^2}{\Lambda}, \quad \alpha_{2,3} = \frac{3\lambda\Lambda + 2\lambda m^2 + 2\Lambda m^2 + m^4}{\lambda\Lambda - m^4}, \quad \alpha_{2,4} = -\frac{m^2}{\lambda}. \quad (12)$$

They must all match cases **1**, **15** and **18** in the Morales-Ramis table [18] for H_h to be integrable:

$$\alpha_{2,i} \in \left\{ \frac{(1+12p)(7+12p)}{72} \right\}_{p \in \mathbb{Z}} \uplus \{p(2p-1)\}_{p \in \mathbb{Z}} \uplus \left\{ \frac{(1+4p)(3+4p)}{8} \right\}_{p \in \mathbb{Z}} =: S_1 \cup S_2 \cup S_3. \quad (13)$$

This property for $\alpha_{2,2} = -\frac{m^2}{\Lambda}$ and $\alpha_{2,4} = -\frac{m^2}{\lambda}$ implies

$$\Lambda, \lambda \in \{\mu_1(p) : p \in \mathbb{Z}\} \uplus \{\mu_2(p) : p \in \mathbb{Z}\} \uplus \{\mu_3(p) : p \in \mathbb{Z}\}, \quad (14)$$

where

$$\mu_1(p) := -\frac{72m^2}{(12p+1)(12p+7)}, \quad \mu_2(p) := -\frac{m^2}{p(2p-1)}, \quad \mu_3(p) := -\frac{8m^2}{(4p+1)(4p+3)}.$$

$\mu_2(p)$ follows exceptional profile (2) for the original FRW Hamiltonian but the other two do not, regardless of p ; hence additional necessary conditions appear for special Hamiltonian H_h .

In order to collate these conditions with the exceptional cases mentioned in [7, §6], let us focus on the remaining non-trivial eigenvalue $\alpha_{2,3}$ in (12). Denote by $R_{i,j}(p, q)$ the value of $\alpha_{2,3}$ whenever $\Lambda = \mu_i(p)$ and $\lambda = \mu_j(q)$, opposite terms following from symmetry by interchanging Λ and λ . A simple value sweep and a simpler limit calculation yields the following cases for which these terms belong to table sets S_2 or S_3 in (13) for some $p, q \in \mathbb{Z}$:

- $R_{1,2}(1, q) = 1 \in S_2$ for every $p \in \mathbb{Z}$.
- $R_{2,2}(p, 1) = R_{2,2}(1, q) = 1 \in S_2$ for every $p, q \in \mathbb{Z}$.
- $R_{2,2}(-1, -1) = 0 \in S_2$.
- $R_{2,3}(1, q) = 1 \in S_2$ for every $p \in \mathbb{Z}$.
- $R_{2,3}(-1, -1) = R_{2,3}(-1, 0) = 21 \in S_2$.
- $R_{2,3}(-8, -1) = R_{2,3}(-8, 0) = \frac{35}{8} \in S_3$ for every $p \in \mathbb{Z}$.

Any other values (including $R_{1,1}(p, q)$ and $R_{3,3}(p, q)$ for any $p, q \in \mathbb{Z}$) do not belong to sets (13). Hence the exceptional values for which non-integrability is not ensured are only those summarised in [7, §6]; namely, those for which the Hamiltonian is already known to be integrable

$$(\Lambda, \lambda) \in \left\{ (-m^2, -m^2), \left(-\frac{m^2}{3}, -\frac{m^2}{3}\right), \left(-\frac{m^2}{3}, -\frac{8m^2}{3}\right), \left(-\frac{m^2}{6}, -\frac{8m^2}{3}\right) \right\}, \quad (15)$$

and those found by Maciejewski et al. and Coelho et al. yielding speculable integrability:

$$(\Lambda, \lambda) \in \left\{ \left(-\frac{m^2}{136}, -\frac{8m^2}{3}\right) \right\} \uplus \{(-m^2, \mu_1(p))\}_{p \in \mathbb{Z}} \uplus \{(-m^2, \mu_2(p))\}_{p \in \mathbb{Z} \setminus \{1\}} \uplus \{(-m^2, \mu_3(p))\}_{p \in \mathbb{Z}}. \quad (16)$$

Our purpose is to study values (16) using higher-order variational equations.

3.2 Particular solutions and first-order variational equations

Let us pave the way for Sections 3.3 and 3.4. The homographic solutions [18] attached to the above Darboux points \mathbf{c} are of the form $\phi = (z\mathbf{c}, \dot{z}\mathbf{c})$ where $\ddot{z} + z^3 = 0$; two such functions are $z_1 = \frac{i\sqrt{2}}{(t-1)}$ and $z_2 = \sqrt{2}\operatorname{sn}(t, i)$. Picking $\mathbf{c} = (\Lambda^{-1}, 0)$, $\phi_i = \sqrt{\frac{1}{\Lambda}}(z_i, 0, \dot{z}_i, 0)$, $i = 1, 2$ are thus solutions to the Hamiltonian.

Each of these two solutions has an asset and a drawback. ϕ_1 contains no special functions and solutions to linearised variational equations $\operatorname{LVE}_{\phi_1}^k$, $k \geq 1$ are easy to compute explicitly, yielding only one non-rational function up to order $k = 5$, namely $\ln(t-1)$, as well as a very simple monodromy matrix up said order. However, the presence of only one singularity aside from ∞ renders the time domain T equal to the Riemann sphere $\mathbb{P}_{\mathbb{C}}^1$ minus two points, whose fundamental group $\pi_1(T, 0)$ is abelian; hence the only obstructions to integrability $\operatorname{Gal}(\operatorname{LVE}_{\phi_1}^k)$ may offer arise from Stokes phenomena at infinity. The fact that the fundamental matrix for $\operatorname{VE}_{\phi_1}^1$ is rational, however, eliminates that possibility and the usefulness of ϕ_1 for our purposes.

We therefore need to use solution $\phi = \phi_2 = \sqrt{\frac{2}{\Lambda}}(\operatorname{sn}(t, i), 0, \operatorname{cn}(t, i) \operatorname{dn}(t, i))$ which does have more than one singularity (infinitely many, for that matter) and yields first-order monodromies which are trivial, hence contained in $\operatorname{Gal}(\operatorname{VE}_{\phi}^1)^\circ$ for Λ belonging to (16) (thereby implying higher-order monodromies will be in $\operatorname{Gal}(\operatorname{LVE}_{\phi}^k)^\circ$), but is harder to manipulate symbolically. A fundamental matrix of $(\operatorname{VE}_{\phi})$ is again defined using Legendre functions on rays exiting 0:

$$\Phi(t) = \begin{pmatrix} f_1 & 0 & f_2 & 0 \\ 0 & g_1 & 0 & g_2 \\ \dot{f}_1 & 0 & \dot{f}_2 & 0 \\ 0 & \dot{g}_1 & 0 & \dot{g}_2 \end{pmatrix}, \quad \begin{cases} f_1 = \sqrt{1-z^4}, \\ f_2 = -\frac{2^{3/4}\pi\sqrt{z}}{\Gamma(-1/4)}P_{3/4}^{-1/4}(\sqrt{1-z^4}), \\ g_1 = \frac{\Gamma(\frac{3}{4})\sqrt{z}}{2^{1/4}}P_{\alpha}^{1/4}(\sqrt{1-z^4}), \\ g_2 = 2^{1/4}\Gamma(\frac{5}{4})\sqrt{z}P_{\alpha}^{-1/4}(\sqrt{1-z^4}), \end{cases} \quad (17)$$

where $z = \operatorname{sn}(t, i)$ and $\alpha = \frac{1}{4}\left(-2 + \sqrt{1 - \frac{8m^2}{\Lambda}}\right)$. An argument akin to the one used in 2.2.1 easily proves that, for $\Lambda = \mu_i(p)$ for $i = 1, 2, 3$, monodromies around poles $\pm t^* = \pm iK(\sqrt{2})$ are

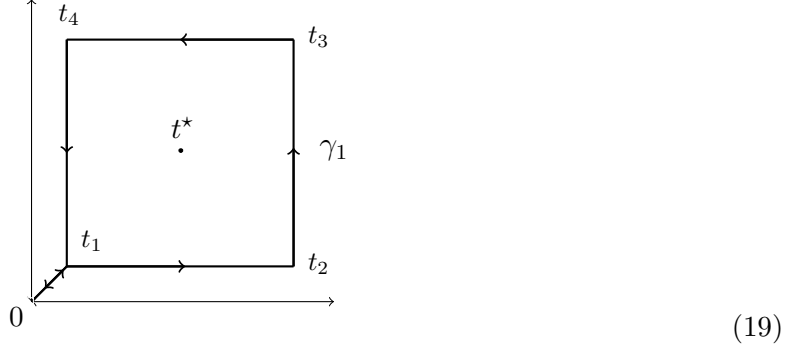
$$M_{\pm}^{\mu_1(p)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}i & 0 & \pm a \mp ai \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2(\pm a - \mp ai)} & 0 & \frac{1}{2} + \frac{1}{2}i \end{pmatrix}, \quad M_{\pm}^{\mu_2(p)} = \operatorname{Id}, \quad M_{\pm}^{\mu_3(p)} = \operatorname{diag}(1, -1, 1, -1), \quad (18)$$

$a = a(p) \neq 0$ being a real number. All three matrices belong to $\operatorname{Gal}(\operatorname{VE}_{\phi}^1)^\circ$; this is obvious in the latter two cases and immediately verifiable in the former by means of a conjugacy and [14, Prop 2.2] since the Zariski closure of the non-trivial block is $\operatorname{SL}_2(\mathbb{C})$. λ makes no intervention until $k = 3$ hence the (non-)triviality of the first two levels depends entirely on the value of Λ . $\Lambda \in \mu_2(\mathbb{Z})$ in all pairs in (16); thus, (18) implies a trivial monodromy for $\operatorname{VE}_{\phi}^1$.

3.3 Numerical evidence

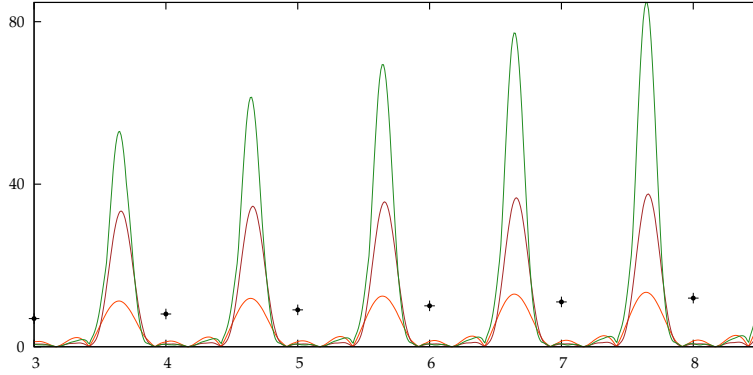
The theoretical framework described in the previous Sections, as well as the difficulty in finding a simple transversal Poincaré section for the orbits of H_h , recommends the computation of numerical monodromies and their commutators as was done in Section 2.1 for the original Hamiltonian. m is set to equal 1 and variational equations are considered along solution $\phi_2 = \sqrt{\frac{2}{\Lambda}}(z(t), 0, \dot{z}(t), 0)$ described in Section 3.2 for the reasons given therein. z is two-periodic and two of its poles are $t^* = iK(\sqrt{2}) \simeq 1.31103 + 1.31103i$ and $-t^*$. Monodromies have been taken

along spoon-shaped paths containing these two poles. A numerical sweep in all simulations seems to indicate that the minimal value of $\|\Phi\|$ for such polygons is γ_1 shown below,

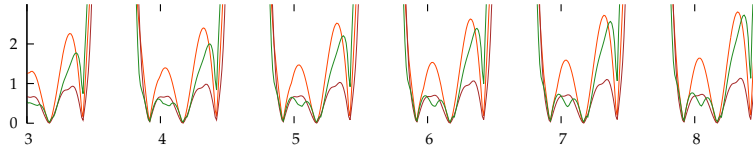


where $t_1 = t^* - 1 - i, t_2 = t^* + 1 - i, t_3 = t^* + 1 + i, t_4 = t^* - 1 + i$, and $\gamma_2 = -\gamma_1$. Set $m = 1$. Monodromies M_{k,γ_i} , $i = 1, 2, k = 1, 2, 3, 4, 5$ and their commutators $C_k = M_{k,\gamma_1}M_{k,\gamma_2} - M_{k,\gamma_2}M_{k,\gamma_1}$ have been numerically simulated for a wealth of values of p for the following cases:

- (i) $\Lambda = \lambda = \mu_i(p)$, $i = 1, 2, 3$, already known non-integrable save for $i = 2, p = 1$, and
- (ii) $\Lambda = -m^2, \lambda = \mu_i(p)$, $i = 1, 2, 3$, i.e. three of the open cases in (16).



(a) $p \in [3, 8]$



(b) Closeup showing non-vanishing of $\|C_{1,2,3}\|_\infty$ for $p \in \mathbb{Z}$

Figure 3: Case $\Lambda = \lambda = \mu_1(p)$: $\sqrt{\|C_1\|_\infty}$, $\frac{\sqrt{\|C_2\|_\infty}}{3}$, $\frac{\sqrt{\|C_3\|_\infty}}{15}$, $\frac{\sqrt{\|C_5\|_\infty}}{15}$, the latter for $p \in \mathbb{N}$ only

Case (i), where visible obstructions to integrability should be no surprise, is interesting in that said obstructions do not necessarily appear at order five as was the case for Hamiltonian (1) with $k \neq 0$ – let us not forget three possible exceptional values μ_1, μ_2, μ_3 are at play, rather than only one μ as in (2). Indeed, for $\Lambda = \lambda = \mu_1(p)$, as seen in Figure 3, obstructions appear at first order already – even though commutators for orders $k = 1, 2, 3, 4$ do vanish periodically at other values shown in 3(b) which of course are of no interest to our study. Case $\Lambda = \lambda = \mu_2$ does show a pattern of monodromy commutation at orders $k = 1, 2, 3, 4$ (for $p \in \mathbb{Z}$ and for other points as well, as seen in Figure 4(b)) followed by non-commutativity at order $k = 5$. Finally, $\Lambda = \lambda = \mu_3(p)$ yields vanishing commutators at $p \in \mathbb{Z}$ for $k = 1, 2, 3, 4, 5$ altogether; which order

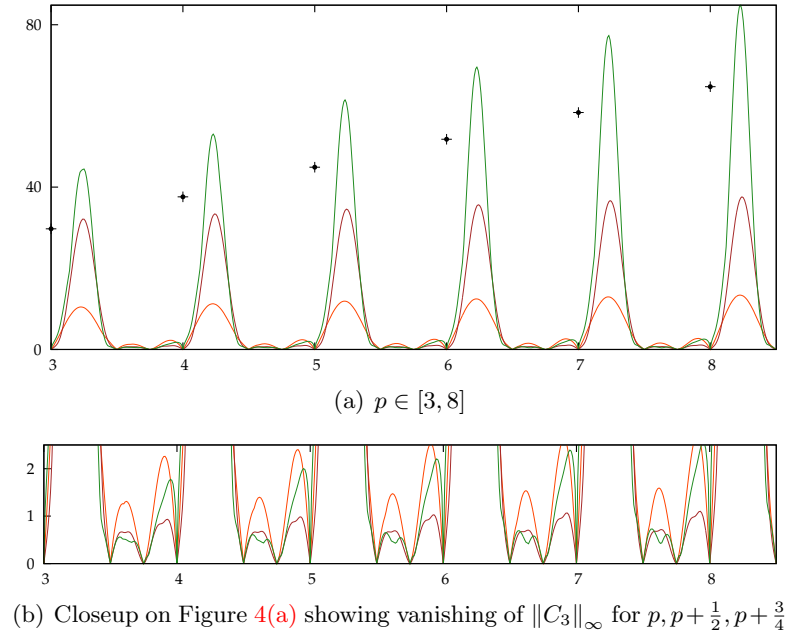


Figure 4: Case $\Lambda = \lambda = \mu_2(p)$: $\sqrt{\|C_1\|_\infty}$, $\frac{\sqrt{\|C_2\|_\infty}}{3}$, $\frac{\sqrt{\|C_3\|_\infty}}{15}$, $\frac{\sqrt{\|C_5\|_\infty}}{15}$, the latter for $p \in \mathbb{N}$ only

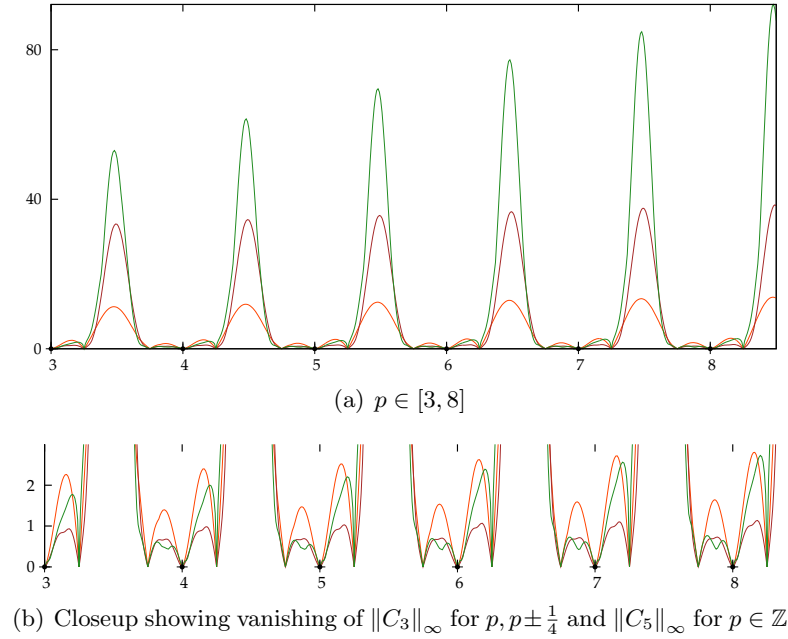


Figure 5: Case $\Lambda = \lambda = \mu_3(p)$: $\sqrt{\|C_1\|_\infty}$, $\frac{\sqrt{\|C_2\|_\infty}}{3}$, $\frac{\sqrt{\|C_3\|_\infty}}{15}$, $\|C_5\|_\infty$, the latter for $p \in \mathbb{N}$ only

k does display non-commutativity in this third case can only be speculated on, although failure to appear in (15), (16) leaves no doubt that H_h is non-integrable for these values of Λ and λ .

Case (ii), based on the open cases for integrability, is paradoxically easier to describe. First-order monodromies for integer p are equal to Id_4 in accordance with the middle case in (18) and the fact that $-m^2 = \mu_2(1)$. In all three cases, $\lambda = \mu_i(p)$, $i = 1, 2, 3$, contrary to what could be inferred from (i), and although monodromies cease to be trivial at $k = 3$, the order of magnitude for commutators at order 4 for $p \in \mathbb{Z}$ remains 10^{-10} – 10^{-9} and obstructions only seem to arise in $k = 5$. This allows us to summarise several figures in a more compact manner, see Figure 6.

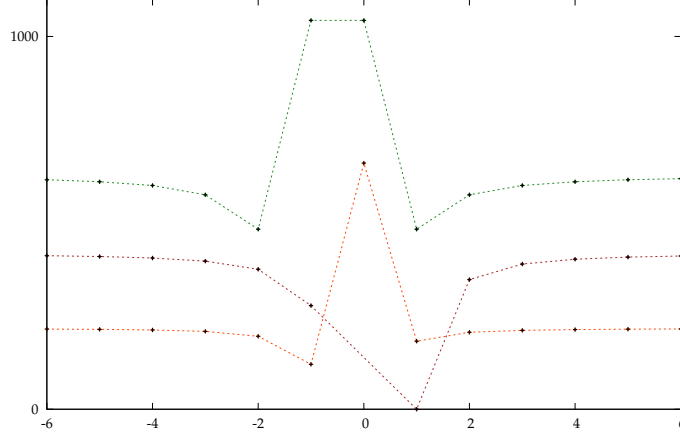


Figure 6: Case $\Lambda = -m^2$: $15(\|C_5\|_\infty)^{1/4}$ for $\lambda = \mu_1(p)$, $2\sqrt{\|C_5\|_\infty}$ for $\lambda = \mu_2(p)$ and $3\sqrt{\|C_5\|_\infty}$ for $\lambda = \mu_3(p)$, in ascending order for $p = -6, \dots, 6$. Note avoidance of $p = 0$ and vanishing at $p = 1$ for $15(\|C_5\|_\infty)^{1/4}$, predictable from the integrable value in (15)

Finally, first pair $(\Lambda, \lambda) = \left(-\frac{m^2}{136}, -\frac{8m^2}{3}\right)$ in (16) yields trivial monodromies up to $k = 5$; higher orders pose a computational challenge in terms of time and will be tackled in future work.

3.4 Condition for non-integrability

Same as in 2.1, logical limitations of the small range of p displayed in 3.3 are compounded by the presence of unchecked numerical propagation. The only foolproof method, therefore is, computing the residue for certain entries of the jet along $\gamma := \gamma_-^{-1}\gamma_+^{-1}\gamma_-\gamma_+$ as was done in 2.2.2.

In all cases, the pattern seems to be the same: non-zero values of the jet K transported along γ are the same as in (7), save for $K_{4,56}$. The one bearing both the largest modulus and the simplest symbolic expression is still $K_{4,36}$, now equal to $\int_{\gamma_-^{-1}\gamma_+^{-1}\gamma_-\gamma_+} A(t) dt$ where

$$A = \frac{20g_1(-3g_1(-4m^6G_{1,3}^2 + \lambda\Lambda G_{2,15}g_1) + 4m^4(G_{1,3}G_{2,15} + 2G_{3,11}g_1)\text{sn}(t, i))}{\Lambda},$$

where g_1, g_2, f_1, f_2 are defined as in (17),

$$G_{1,3} = -f_1F_{1,5} + F_{1,4}f_2, \quad G_{2,15} = \frac{6F_{2,29}g_1}{\Lambda} + F_{2,30}g_2, \quad G_{3,11} = f_2F_{3,21} + f_1F_{3,22},$$

and quadratures arising from $\text{LVE}_\phi^{2,3,4}$ closely resemble those in 2.2.2

$$\begin{aligned} F_{1,5}(t) &= \int f_2(\tau) g_1(\tau)^2 \text{sn}(\tau, i) d\tau \\ F_{1,4}(t) &= \int f_1(\tau) g_1(\tau)^2 \text{sn}(\tau, i) d\tau \\ F_{3,21}(t) &= \int f_1(\tau) \left(3m^2 G_{1,3}(\tau) g_1(\tau)^2 - 18m^2 G_{1,3}(\tau)^2 \text{sn}(\tau, i) + G_{2,15}(\tau) g_1(\tau) \text{sn}(\tau, i) \right) d\tau \\ F_{3,22}(t) &= \int f_2(\tau) \left(-3m^2 G_{1,3}(\tau) g_1(\tau)^2 + \left(18m^2 G_{1,3}(\tau)^2 - G_{2,15}(\tau) g_1(\tau) \right) \text{sn}(\tau, i) \right) d\tau \\ F_{2,29}(t) &= \int g_2(\tau) \left(\lambda\Lambda g_1(\tau)^3 - 4m^4 G_{1,3}(\tau) g_1(\tau) \text{sn}(\tau, i) \right) d\tau \\ F_{2,30}(t) &= \int \left(-6\lambda g_1(\tau)^4 + \frac{24m^4 G_{1,3}(\tau) g_1(\tau)^2 \text{sn}(\tau, i)}{\Lambda} \right) d\tau \end{aligned}$$

Sections 3.2 and 3.4 therefore build up the proof for the following:

Proposition 3.4.1. *For any values of (Λ, λ) for which $K_{4,36} \neq 0$, H_h is non-integrable. \square*

An educated guess in view of Figure 6 and the asymmetry of μ_i with respect to $p = 0$ is:

Conjecture 3.4.2. *$K_{4,36} \neq 0$ if $\Lambda = -m^2$ and $\lambda = \mu_i(p)$, $i = 1, 2, 3$, for infinitely many $p \in \mathbb{Z}$.*

Needless to say, simulations done so far preclude neither the existence of values of p for which $K_{4,36}$ does equal zero, nor the possible misleading effect of numerical errors on higher variational orders. Forthcoming work in progress will bear the bulk of such tasks by proving Conjectures 2.2.2 and, especially, 3.4.2.

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