On non-smooth pitchfork bifurcations in invertible quasi-periodically forced 1-D maps

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In this note we revisit an example introduced by T. Jäger in which a Strange Non-chaotic Attractor seems to appear during a pitchfork bifurcation of invariant curves in a quasi-periodically forced 1-d map. In this example, it is remarkable that the map is invertible and, hence, the invariant curves are always reducible.

In the first part of the paper we give a numerical description (based on a precise computation of invariant curves and Lyapunov exponents) of the phenomenon. The second part consists in a preliminary study of the phenomenon, in which we prove that an analytic self-symmetric invariant curve is persistent under perturbations.

Keywords: Symmetric invariant curves; Newton method; reducibility.

AMS Subject Classification: 37C55; 37G40; 37C60.

1. Introduction

In this note we focus on the existence of pitchfork bifurcations of invariant curves in a quasi-periodically forced model introduced by T. Jäger in [5]. Here we use a rescaled version of this map, namely

$$\left\{ \begin{array}{l} x_{n+1} = \arctan(ax_n) + b\sin\theta_n, \\ \theta_{n+1} = \theta_n + \omega, \end{array} \right. \tag{1}$$

where $\theta \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, ω satisfies a Diophantine condition and $a > 0$. To simplify notation, let us define $\varphi(x,\theta) = \arctan(ax) + b\sin\theta$. Note that this dynamical system satisfies several important properties:

(a) For each θ, $\partial_x\varphi(x,\theta) > 0$ and hence $\varphi(\,\cdot\,,\theta)$ is invertible and (1) has monotone fibre maps: $\varphi(x,\theta) < \varphi(y,\theta)$ if $x < y$.
(b) For each θ, $\varphi(\,\cdot\,,\theta)$ has negative Schwarzian derivative: $\frac{\partial_x^3\varphi}{\partial_x\varphi} - \frac{3}{2}(\frac{\partial_x^2\varphi}{\partial_x\varphi})^2 < 0$.
(c) The map (1) is invariant by the symmetry $S : (x,\theta) \mapsto (-x, \theta + \pi)$.
(d) $\varphi(\mathbb{R},\mathbb{T})$ is bounded.

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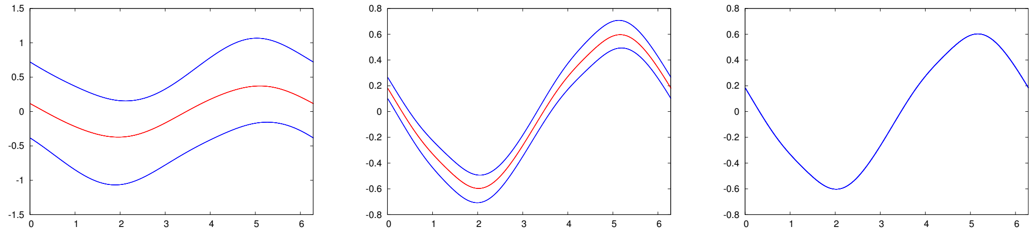


Figure 1. Invariant curves for $a = 1.25$ and $b = 0.75$ (left), $b = 1.14$ (centre) and $b = 1.15$ (right). The horizontal axis shows the angle θ and the vertical axis the value x. Attracting curves are displayed in blue and repelling ones in red.

We stress that property (a) is not satisfied by other classical examples of quasi-periodically forced 1-D maps such as the quasi-periodically forced logistic map.

There are some previous results for this type of maps. For instance, in [5], corollary 4.3, it is proved that one of the following is true:

(1) There exists an invariant graph self-symmetric by S which is a (continuous) curve if its Lyapunov exponent is negative.
(2) There exist three invariant graphs, one of them is self-symmetric by S and the other two are "mirror" images under S. If one of these three graphs is continuous, then so are the other two. If the graphs are not continuous, the essential closure (see definition 2.4 in [5]) is the same for all of them.

Direct numerical simulations show evidence that for some range of values of parameters, a pitchfork bifurcation occurs (see Figure 1). It is remarkable that, for other ranges of parameters, a strange set seems to appear at the bifurcation point (see Figure 2). It is known that sometimes invariant curves can be dramatically wrinkled so that they look as a strange set when they are still smooth (see [9] for concrete examples). In this case, careful numerical magnifications of the attracting sets in Figure 2 suggest that all these sets are in fact smooth invariant curves. Still, it is an open question (that cannot be solved by pure numerical simulations) if the merging of the branches of the pitchfork bifurcations takes place on a non-smooth invariant set. We intend with this paper to provide some results and numerical tools which are able to offer an insight into this question.

The system has two parameters (a and b), and the pitchfork bifurcation appears when a is fixed to a moderate value and b moves on a suitable range. In this bifurcation, there are three invariant curves (two stable, one being the image of the other by the symmetry S, and one unstable which is self-symmetric) that merge and become a stable self-symmetric invariant curve, as seen in Figure 1. Now let us focus on the self-symmetric invariant curve that goes from unstable to stable when the bifurcation takes place. As long as it exists, this self-symmetric curve can be labelled by the parameters (a, b) so let us denote by $\Lambda(a, b)$ its Lyapunov exponent: if $x_{a,b} : \mathbb{T} \to \mathbb{R}$ is a parametrization of this curve, then

$$\Lambda(a, b) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{\partial \varphi}{\partial x}(x_{a,b}(\theta), \theta) \right| \, d\theta,$$

where we recall that $\varphi(x, \theta) = \arctan(ax) + b \sin\theta$. As usual, we will refer to the value $\lambda = \exp \Lambda$ as Lyapunov multiplier.

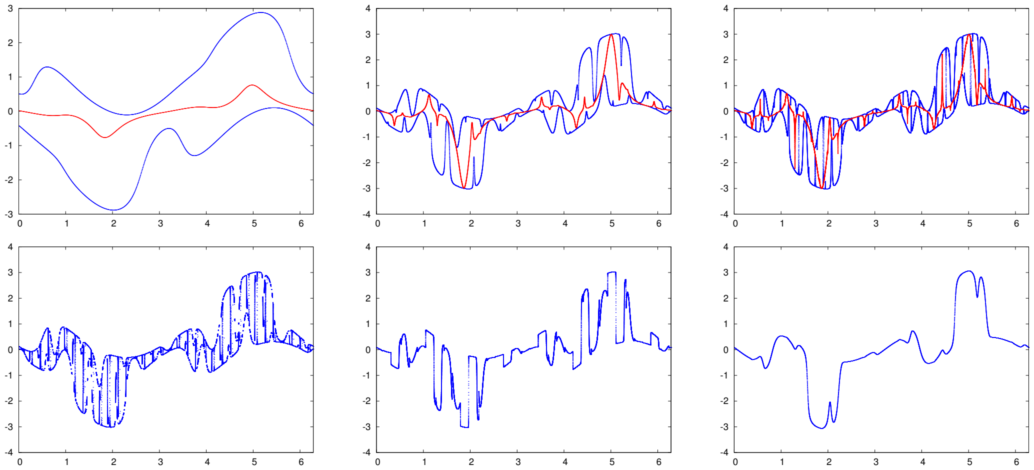


Figure 2. Invariant curves for $a = 6.8$. Upper row: $b = 1.62$ (left), $b = 1.82$ (centre) and $b = 1.8204$ (right). Lower row: $b = 1.8205$ (left), $b = 1.84$ (centre) and $b = 2$ (right). Attracting curves are displayed in blue and repelling ones in red. The axis are the same as in Figure 1.

Hence, for a given (and small) value of a, we call $b(a)$ the value of b corresponding to the merging of the three invariant curves, such that for the parameter values $(a, b(a))$ the map only has a self-symmetric invariant curve with zero Lyapunov exponent. Now, let us focus on the curve $(a, b(a))$ of the parameter space that is well defined as long as there is a (smooth) pitchfork bifurcation. To simplify future computations, let us parametrize this curve using the arclength parameter of the (a, b) plane, so we write it as $s \mapsto \Gamma_1(s) = (a(s), b(s))$, being s the arclength parameter. Moreover, we can extend this definition to self-symmetric invariant curves corresponding to parameters (a, b) such that the Lyapunov multiplier of the curve is a given value λ. Therefore, we consider the curve on the parameter space (a, b), that we write as $s \mapsto \Gamma_\lambda(s) = (a(s), b(s))$, corresponding to parameters for which the self-symmetric invariant curve has Lyapunov multiplier λ.

In Section 2 we compute the curves of the parameter space defined by the image of Γ_λ, for different values of λ. The goal is to display the regions of the parameter space for which there are regular invariant curves, to isolate regions on which strange invariant sets could appear. Section 3 is devoted to the proof of the persistence of self-symmetric invariant curves. This section also serves as a theoretical support for the computational methods of Section 2.

2. Numerical continuation of invariant curves

Here we focus on the numerical computation of invariant curves with a prescribed Lyapunov exponent. Let us start by explaining the computation of invariant curves (the numerical method we will use here is based on the method developed in [7], which is based on the results in [8]). The considered dynamical system is

$$\begin{cases} x_{n+1} = \arctan(a\, x_n) + b\, \sin\theta_n, \\ \theta_{n+1} = \theta_n + \omega, \end{cases}$$

and we stress that the computational method is valid for general 1D maps with quasi-periodic forcing. The goal is to compute a smooth curve $\theta \mapsto x(\theta)$ such that, for all $\theta \in \mathbb{T}$, it satisfies $x(\theta+\omega) = \arctan(ax(\theta)) + b\sin\theta$. The curve $\theta \mapsto x(\theta)$ will be represented by its (truncated) Fourier series, and the truncation value will be selected to have a prescribed accuracy (this will be discussed later with more detail). Note that the self-symmetric condition implies that the average of the curve must be zero, so we will not include the first Fourier coefficient in the expansions.

2.1. *Computation of an invariant curve*

The computational method is based on a Newton iteration in the space of Fourier coefficients. As in any Newton procedure, we start from a set of Fourier coefficients that correspond to an approximate invariant curve $\theta \mapsto x_0(\theta)$. By approximate, we mean that a norm of the map

$$\theta \mapsto r_0(\theta) = x_0(\theta+\omega) - \arctan(ax_0(\theta)) - b\sin\theta,$$

is small. As x_0 is given by a (finite) set of Fourier coefficients, the Fourier coefficients of the composition $\arctan(ax_0(\theta))$ are obtained by computing a (finite) table of values of x_0, $\{x_0(\theta_j)\}_j$, and then applying a Fast Fourier Transform (FFT) to the values $\{\arctan(x_0(\theta_j))\}_j$. Hence, given the Fourier coefficients of x_0, it is not difficult to compute the coefficients of r_0, with a complexity of $O(N\log N)$, where N denotes the number of Fourier coefficients. The next step is to look for a map h such that $x_1 = x_0 + h$ is a better approximation to the invariant curve. To find h, let us linearise at x_0,

$$\begin{aligned}
\theta \mapsto r_1(\theta) &= x_1(\theta+\omega) - \arctan(ax_1(\theta)) - b\sin\theta,\\
&= x_0(\theta+\omega) + h(\theta+\omega) - \arctan(a(x_0(\theta)+h(\theta))) - b\sin\theta,\\
&= h(\theta+\omega) - p(\theta)h(\theta) - q(\theta) + O_2(|h(\theta)|),
\end{aligned}$$

where the term O_2 denotes the Taylor remainder, and

$$p(\theta) = \frac{a}{1+(ax_0(\theta))^2}, \qquad q(\theta) = \arctan(ax_0(\theta)) + b\sin\theta - x_0(\theta+\omega).$$

Skipping the second order term $O_2(h)$, we have that h satisfies the affine equation

$$h(\theta+\omega) = p(\theta)h(\theta) + q(\theta). \tag{2}$$

To find h we use that the linear dynamical system

$$\begin{cases} h_{n+1} = p(\theta_n)h_n, \\ \theta_{n+1} = \theta_n + \omega, \end{cases} \tag{3}$$

is reducible: there exists a change of coordinates of the form $h = c(\theta)y$ (that is, linear in h) such that the transformed system is reduced to constant coefficients,

$$\begin{cases} y_{n+1} = \lambda y_n, \\ \theta_{n+1} = \theta_n + \omega, \end{cases}$$

where $\lambda = p(\theta)c(\theta)/c(\theta+\omega)$ does not depend on θ. Not all the linear quasi-periodic systems are reducible, but the one considered here is (the reason is that p is smooth and has no zeros, see [9] for the details). The computation of the reducing transformation $c(\theta)$ follows from

$$\log \lambda = \log p(\theta) + \log c(\theta) - \log c(\theta+\omega), \quad \forall \theta \in \mathbb{T}. \tag{4}$$

If we denote as $\{p_k\}_k$ the Fourier coefficients of $\log \circ p$ and $\{d_k\}_k$ the ones of $\log \circ c$,

$$p_k := \frac{1}{2\pi}\int_0^{2\pi} \log(p(z))e^{-ikz}\,dz, \quad c_k := \frac{1}{2\pi}\int_0^{2\pi} \log(c(z))e^{-ikz}\,dz.$$

then (4) can be easily solved by expanding it in Fourier series: The coefficient c_0 is undetermined (so we take it equal to zero), $\log\lambda = p_0$ (note that p_0 is the Lyapunov exponent of (3)), and

$$c_k = \frac{p_k}{e^{ik\omega}-1}, \quad k \neq 0. \tag{5}$$

Once c has been found, we can apply the change $h = c(\theta)y$ to the affine equation (2) so that it takes the form

$$y(\theta+\omega) = \lambda y(\theta) + \widehat{q}(\theta),$$

where $\widehat{q}(\theta) = q(\theta)/c(\theta)$. Note that this last equation can be easily solved using the Fourier coefficients of the involved functions:

$$y_k = \frac{\widehat{q}_k}{e^{ik\omega}-\lambda}, \quad k \neq 0,$$

and $y_0 = 0$ (the case $k = 0$ follows from the symmetries of the problem, see Section 3 for the details). Then, we recover $h(\theta) = c(\theta)y(\theta)$ and we can complete the step of the Newton method, $x_1 = x_0 + h$.

Note that, in this process, the amount of computations is very low. An invariant curve is stored as an array (of length N) of Fourier coefficients or as an equispaced table of values (again of length N), the conversion from one format to the other can be done in a fast and efficient way using FFTs, with a complexity of $O(N\log N)$ operations. Each of the previous steps can be done very efficiently (in $O(N)$ operations) if the maps are stored in the right format (for instance, the computation of a table of values of $p(\theta) = a/(1+(ax_0(\theta))^2)$ is immediate if one has a table of values of x_0, and the computation of c through expression (5) is also trivial if one has the Fourier coefficients of p). Hence, a single step of Newton method requires $O(N\log N)$ arithmetic operations and $O(N)$ computer memory. This allows to use large values of N (for instance, larger than 10^6) which is required to approximate wrinkled invariant curves.

2.1.1. Error control

Error estimates can be easily obtained in the following form. Given an approximation $\widetilde{x}$ to an invariant curve, let us define $\widetilde{r}(\theta) = \widetilde{x}(\theta+\omega) - \arctan(a\widetilde{x}(\theta)) - b\sin\theta$. Then, we can estimate the value $\max_\theta |\widetilde{r}(\theta)|$ by evaluating $\widetilde{r}$ in a mesh finer than the one used to compute $\widetilde{x}$. If this value is larger than a prescribed threshold (say, 10^{-10}), then the value of N is enlarged and the computation is repeated with this new N. For more details about these error estimates, see [3, 6].

2.2. *Continuation*

We are not interested in the computation of a single invariant curve but in the computation of families of self-symmetric invariant curves. More concretely, let Ω to be an open set of the parameters (a, b) for which there exists a self-symmetric invariant curve (as we will se in Section 3, if a self-symmetric invariant curve exists for a value (a, b), it exists for a neighbourhood of it). We are interested in curves of the parameter space, $s \mapsto \Gamma_\lambda(s) = (a(s), b(s))$, corresponding to parameters for which the self-symmetric invariant curve has Lyapunov multiplier λ.

As it has been mentioned before, we will focus on the system,

$$\begin{cases} x_{n+1} = \arctan(a\, x_n) + b \sin\theta_n, \\ \theta_{n+1} = \theta_n + \omega, \end{cases}$$

where $\omega = (\sqrt{5} - 1)\pi$ is the golden mean. Our goal is to provide a numerical approximation to the coefficient $\boldsymbol{s} = (s_1, s_2, \ldots)$ of the Fourier series of a self-symmetric invariant curve with a given Lyapunov exponent,

$$x(\theta, \boldsymbol{s}) = \sum_{k\in\mathbb{N}\setminus\{0\}} s_k e^{(2k-1)\theta\, i} + \sum_{k\in\mathbb{N}\setminus\{0\}} \overline{s_k} e^{(1-2k)\theta\, i}. \tag{6}$$

This function must verify the equation for being an invariant curve,

$$F(\theta, a, b, \boldsymbol{s}) = x(\theta + \omega, \boldsymbol{s}) - \arctan(ax(\theta, \boldsymbol{s})) - b \sin\theta = 0 \quad \text{for all } \theta \in \mathbb{T}, \tag{7}$$

and the Lyapunov multiplier equal to a given value λ. The approximation of the invariant curve can be calculated truncating the Fourier series and finding the solution of the system given by equation (7) for values of θ on an equally distributed mesh, with an extra equation asking the Lyapunov multiplier to be equal to λ. This is equivalent to find the zeros of the following function,

$$(\boldsymbol{s}, a, b) \mapsto (F(\,\cdot\,, a, b, \boldsymbol{s}), L(a, b, \boldsymbol{s}) - \log\lambda), \tag{8}$$

where now $\boldsymbol{s}$ denotes a truncated list of $2m$ coefficients, and L is the approximation to the Lyapunov exponent obtained by numerical integration (using the trapezoidal rule) of the log of the derivative of the map along the curve,

$$L(a, b, \boldsymbol{s}) = \frac{\pi}{m} \sum_{j=0}^{2m-1} \log \left| \frac{a}{1 + a^2\, x(\theta_j, \boldsymbol{s})^2} \right|.$$

Newton method can be used to find numerically the zeros of (8). For the sake of efficiency, we will use the reducibility of the invariant curves: Note that as we approximate the curve by $N = 2m$ Fourier coefficients, the linear system that appears at each step of the Newton method is of dimension N, which is prohibitive for large values of N. On the other hand, when the linear behaviour around the (approximated) curve is reduced to constant coefficients, this linear system has a diagonal matrix, which is solved in only N operations.

As example of continuation, let us fix the parameter a and let us take b as the continuation parameter. Hence, the unknowns are the parameter b and the $2n$ coefficients $\boldsymbol{s}$. At Newton step n, the correction of x_n and b_n are denoted by $h_x = x_{n+1} - x_n$ and $h_b = b_{n+1} - b_n$, and they satisfy

$$
\begin{aligned}
h_x(\theta+\omega) - \frac{a h_x(\theta)}{1+(a x_n(\theta))^2} - h_b \sin\theta &= -x_n(\theta+\omega) + \arctan(a x_n(\theta)) + b_n \sin\theta, \\
\frac{\pi}{m}\sum_{j=0}^{2m-1} \frac{-2a^2 x_n(\theta_j)}{1+(a x_n(\theta_j))^2} h_x(\theta_j) &= \log\lambda - L(a, b_n, x_n).
\end{aligned}
$$

We can use Lemma 3.7 to obtain a function c such that it reduces the linear part to constant coefficients, $\frac{a}{1+(a\, x_n(\theta))^2} c(\theta) = \lambda_n c(\theta+\omega)$. Defining $h_y(\theta) = h_x(\theta)/c(\theta)$, the previous system becomes

$$
h_y(\theta+\omega) - \lambda_n h_y(\theta) - \frac{h_b \sin\theta}{c(\theta+\omega)} = \frac{-x_n(\theta+\omega) + a\arctan(a x_n(\theta)) + b_n \sin\theta}{c(\theta+\omega)}, \tag{9a}
$$

$$
\frac{\pi}{m}\sum_{j=0}^{2m-1} \frac{-2a^2 x_n(\theta_j) c(\theta_j)}{1+(a x_n(\theta_j))^2} h_y(\theta_j) = \log\lambda - L(a, b_n, x_n). \tag{9b}
$$

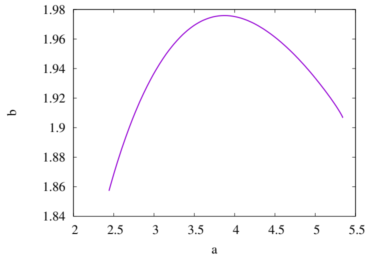
The following step consists in isolating the value h_b. For this, the equation (9a) is decoupled in two equations with solution r_b and r_0 such that $h_b r_b + r_0$ is the solution of (9a) for any value h_b by the superposition principle,

$$
\begin{aligned}
r_b(\theta+\omega) - \lambda_n r_b(\theta) &= \frac{\sin\theta}{c(\theta+\omega)}, \\
r_0(\theta+\omega) - \lambda_n r_0(\theta) &= \frac{-x_n(\theta+\omega) + a\arctan(a x_n(\theta)) + b_n \sin\theta}{c(\theta+\omega)}.
\end{aligned}
$$

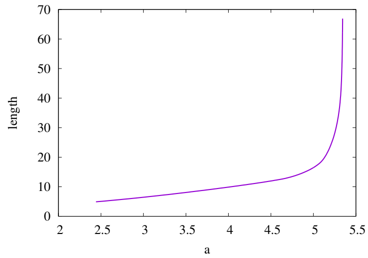
These equations can be approximated by the same schema as Lemma 3.9 to obtain a good numerical approximation to their solution. Substituting $h_b r_b + r_0$ in the integral (9b), we obtain a linear equation for h_b so that it can be isolated,

$$
h_b = \frac{\log\lambda - L(a, b_n, x_n) - \frac{\pi}{m}\sum_{j=0}^{2m-1} z(\theta_j) r_0(\theta_j)}{\frac{\pi}{m}\sum_{j=0}^{2m-1} z(\theta_j) r_b(\theta_j)},
$$

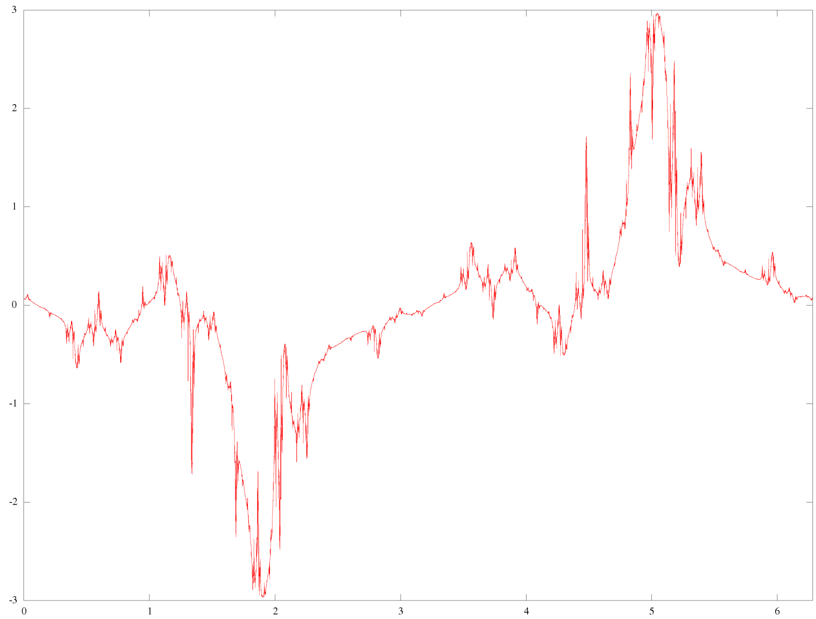
where $z(\theta_j) = \frac{-2a^2 x_n(\theta_j) c(\theta_j)}{1+(a x_n(\theta_j))^2}$. This value h_b and $h_y(\theta) = h_b r_b(\theta) + r_0$ are the solution of equations (9). All these ideas above can be implemented applying a Fast Fourier Transform in a sequential way. For this algorithm only some vectors need to be stored, being the memory used of order the size of the terms in the Fourier series. In contrast solving directly the zeros of the function (8) with a Newton method, a complete matrix of size the number of terms in the Fourier series must be entered in memory, being quite memory-demanding for this reason.



(a) Values of (a, b) corresponding to invariant curves with zero Lyapunov exponent.



(b) Length of the previous invariant curves w.r.t. the value of a.



(c) The last computed invariant curves with zero Lyapunov exponent. The parameter values are $a = 5.348847$ and $b = 1.905990$.

Figure 3. Summary of the computation of self-symmetric invariant curves with zero Lyapunov exponent.

The continuation of invariant curves with zero Lyapunov exponent has been performed and the results are displayed on Figure 3(a), being this continuation a regular curve on the parameter plane (a, b). The situation changes at the endpoint of this curve where algorithm is stopped since continuation cannot be performed since the error is too big. Note that the length of the invariant curves during this continuation seems to have a vertical asymptote when approaching this point, see Figure 3(b). The last computed invariant curve with zero Lyapunov exponent is shown in Figure 3(c), where nearly 49×10^6 Fourier coefficients have been used in the computation. Although the curve is still analytic, it shows a graph very similar to a typical non-rectifiable curve. Consequently, continuation of analytic self-symmetric invariant curves is ended and an invariant symmetric curve with unbounded variation is suggested to appear at the limit. Figure 4 shows curves of parameters (a, b) for which there is a self-symmetric invariant curve with a given (not necessarily zero) Lyapunov exponent. The zone without curves is due to the fact that, when the error is larger than 10^{-7}, the computation is stopped.

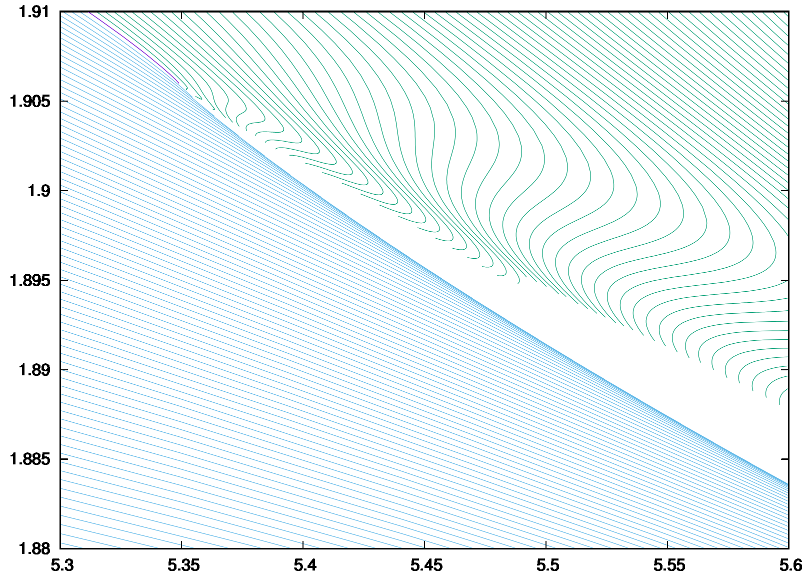


Figure 4. Plot in the space of parameters (a, b). The blue curves at the bottom part of the plot correspond to values of (a, b) with an invariant self-symmetric curve with positive Lyapunov exponent. The green curves in the upper part correspond to invariant self-symmetric curves with negative Lyapunov exponent. The short magenta curve in between contain the values of (a, b) for which the system has a self-symmetric invariant curve with zero Lyapunov exponent.

3. Persistence of self-symmetric invariant curves

In this Section we focus on discrete dynamical systems which can be written as an odd function $\phi = \phi(x)$ plus a quasi-periodic forcing $\psi = \psi(\theta)$ with the symmetry $\psi(\theta + \pi) = -\psi(\theta)$,

$$\begin{cases} x_{n+1} = \phi(x_n) + \psi(\theta_n), \\ \theta_{n+1} = \theta_n + \omega, \end{cases} \tag{10}$$

where ω satisfies a Diophantine condition: If γ and τ are positive numbers, we assume that ω verifies that

$$|k\omega - 2\pi\, m| \geq \frac{\gamma}{|k|^\tau}, \qquad \forall (m, k) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}). \tag{11}$$

An analytic function $\theta \in \mathbb{T} \mapsto x(\theta) \in \mathbb{R}$ is said to be an invariant curve of (10) if

$$x(\theta + \omega) = \phi(x(\theta)) + \psi(\theta), \quad \text{for all } \theta \in \mathbb{T}. \tag{12}$$

We introduce the translation operator T_ω defined as $T_\omega(x)(\theta) = x(\theta+\omega)$. With this notation, invariant curves satisfy the functional equation

$$T_\omega x = \phi \circ x + \psi.$$

This system is invariant with respect to the symmetry $S : (x, \theta) \mapsto (-x, \theta + \pi)$. Hence, if $x(\theta)$ is a invariant curve of (10), then $-x(\theta+\pi)$ is also an invariant curve. A self-symmetric invariant curve of (10) with respect to the symmetry of the system

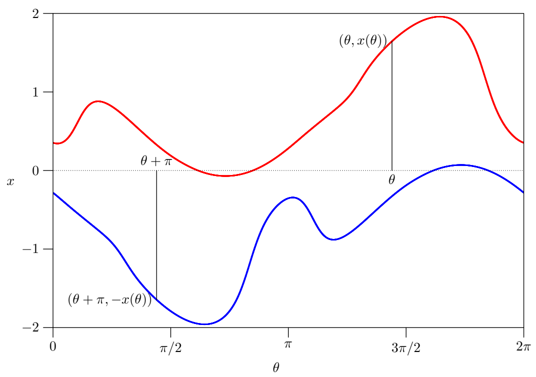


Figure 5. The symmetry of the system $(\theta, x) \mapsto (\theta+\pi, -x)$ implies that given an invariant curve, there is an invariant curve associated by means of this transformation.

satisfies that $x(\theta) = -x(\theta+\pi)$ for all $\theta \in \mathbb{T}$.

Several spaces of analytic functions are needed. If $\rho > 0$, the space $\mathscr{H}_\rho$ denotes the set of real analytic and 2π-periodic functions defined on the (closed) complex strip $K_\rho := \{z \in \mathbb{C} \text{ s.t. } |\mathrm{Im}\,(z)\,| \le \rho\}$, endowed with the sup norm: if $x \in \mathscr{H}_\rho$, $\|x\|_\rho = \max\limits_{z\in K_\rho} |x(z)|$. Other notation for spaces are given by $\mathscr{S}_\rho^-$ which denotes the set of real analytic functions f such that $f(z+\pi) = -f(z)$ for all $z \in K_\rho$, and $\mathscr{S}_\rho^+$ denotes the set of functions π-periodic on K_ρ.

The results in this section are based on KAM methods, see [1, 2, 4, 8].

Theorem 3.1. *Let R, I and ρ be positive numbers and $U := (-R, R) + i(-I, I)$. Let ω be a positive number which fulfills the Diophantine condition (11). Let ϕ be a odd analytic function on $\overline{U}$ such that the derivative is not a negative real number, i.e., $\phi'(\overline{U}) \subseteq \mathbb{C}\backslash]-\infty, 0]$ and $\phi(-z) = -\phi(z)$ for all $z \in \overline{U}$. Let ψ belong to $\mathscr{S}_\rho^-$.*

Then, there exists a constant $C_\omega^{[3.1]}$ such that if there exists $x_0 \in \mathscr{S}_\rho^-$ with $x_0(K_\rho) \subseteq U$ and δ with $0 < \delta < \rho$ such that

$$\|T_\omega x_0 - \phi\circ x_0 - \psi\|_\rho < d\frac{\exp(-\delta^{-\tau-1}C_\omega^{[3.1]}(1+\|\log\circ\phi'\|_{\mathscr{C}^0(\overline{U})}))}{1+\|\phi''\|_{\mathscr{C}^0(\overline{U})}} \tag{13}$$

where

$$d := \frac{1}{2}\min\{R - \max_{z\in K_\rho}|\mathrm{Re}(x_0(z))|, I - \max_{z\in K_\rho}|\mathrm{Im}(x_0(z))|, 1\},$$

then there exists $x \in \mathscr{S}_{\rho_0-\delta}$ with $T_\omega x = \phi\circ x + \psi$.

Here we focus on the particular case $\phi(x) = f(ax)$ and $\psi = bg(\theta)$, where $a > 0$ and b are parameters.

Corollary 3.2. *Let f be an analytic function on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, $f'(x) > 0$ and $f(-x) = -f(x)$. Let g be an analytic function on $\mathbb{R}$ which verifies, for all $\theta \in \mathbb{T}$, $g(\theta+\pi) = -g(\theta)$. Moreover, we assume that ω is a Diophantine number.*

If there exist x_0 analytic function on $\mathbb{R}$ and values $a_0 \in \mathbb{R}^+ \setminus \{0\}$ and $b_0 \in \mathbb{R}$ such that $x_0(\theta+\omega) = f(a_0\, x_0(\theta)) + b_0\, g(\theta)$ and $x_0(\theta+\pi) = -x_0(\theta)$, then there exist a *neighbourhood of* (a_0, b_0) *and* $\rho > 0$ *such that for all* (a, b) *in that neighbourhood there exist an analytic function* $x_{a,b} \in \mathscr{S}_\rho^-$ *which verifies*

$$x_{a,b}(z+\omega) = f(a\, x_{a,b}(z)) + b\, g(z)$$

for all $z \in K_\rho$.

According to this corollary, invariant analytic curves are persistent under some symmetric conditions. It is known (see [9] for a detailed discussion) that invariant curves with non-zero Lyapunov exponent are persistent in the case of invertible skew-product. The main point of the present result is that it does not require any condition on the Lyapunov exponent of the curve so, in particular, it can be zero.

It is possible to prove that the solutions provided by the corollary depend on the parameters a and b in a continuous way, but we have consider that this is a mere technicality whose proof can be easily done by the reader.

The scheme of the proof follows closely the numerical method used in Section 2. Or, in other words, the numerical method can be seen as a computer implementation of the proof of Theorem 3.1. In Section 3.1 we give estimates for the solution of the linearized equation that appears at each step of the Newton method, and in Section 3.2 is devoted to the convergence of the Newton scheme.

3.1. *Affine skew products*

This section is devoted to the study of the affine skew products that appear at each Newton iteration. These skew products are given by the equations

$$\begin{cases} x_{n+1} = p(\theta_n)x_n + q(\theta_n) \\ \theta_{n+1} = \theta_n + \omega \end{cases} \tag{14}$$

where ω is Diophantine (11), q belongs to $\mathscr{S}_\rho^-$ and p is a 2π periodic function and belongs to the set of nonnegative functions on K_ρ in $\mathscr{H}_\rho$, i.e.

$$\mathscr{P}_\rho := \{p \in \mathscr{H}_\rho \,:\, p(K_\rho) \subseteq \mathbb{C}\backslash]-\infty, 0]\}.$$

In fact, in the context of this paper, as $p(\theta) = \phi'(x(\theta))$ where ϕ is real analytic, odd, and x is self-symmetric ($x(\theta+\pi) = -x(\theta)$), we have that p is a π periodic function.

To simplify the notation most of the results are written using operators. We recall that the translation operator $T_\omega : x \in \mathscr{H}_\rho \mapsto \mathscr{H}_\rho$ is defined by $T_\omega x(z) = x(z+\omega)$ for all $z \in K_\rho$. The invariant curves of (14) verify the equation $T_\omega x = p \cdot x + q$ where $p \in \mathscr{P}_\rho \cap \mathscr{S}_\rho^+$, $q \in \mathscr{S}_\rho^-$ and ω Diophantine. As usual, the supremum norm of functions on K_ρ will be denoted by $\|\cdot\|_\rho$.

During the following sections, several constants mainly depending on the frequency ω are defined, in order to be able to trace the constants in inequalities, the superscripts are indicating the lemma where the constant is defined and the subscripts point out the dependencies of the constant.

The following are technical lemmas that are similar to the ones in Section 4 of [1] used to prove KAM theory. They are used in a similar way for the proof of the main result that the one done by Arnold in 1963, but some of them are more specific to prove the theorem presented here (see also [8]). The proof of these lemmas can be found in the appendices

LEMMA 3.3. *If f real 2π–periodic analytic function on K_ρ, then*

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(z)e^{-ikz}\,dz\right| \le \|f\|_\rho e^{-|k|\rho}.$$

LEMMA 3.4. *Let $\rho > 0$, $\delta \in (0,\rho)$ and $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence of analytic functions on $K_{\rho-\delta}$. If for all δ there exists a sequence $\{M_k(\delta)\}_{k\in\mathbb{Z}}$ of positive real numbers with $\|f_k\|_{\rho-\delta} \le M_k(\delta)$ and $\sum_{k\in\mathbb{Z}} M_k(\delta)$ is convergent, then $f(z) = \sum_{k\in\mathbb{Z}} f_k(z)$ converges to an analytic function on the interior of K_ρ, $\mathring{K}_\rho$ and $\|f\|_{\rho-\delta} \le \sum_{k\in\mathbb{Z}} M_k(\delta)$ for all $\delta \in (0,\rho)$.*

LEMMA 3.5. *If $\omega \in \mathbb{R}$ verifies the Diophantine condition* (11) *and $\lambda > 0$, then*

$$\frac{1}{|e^{ik\omega} - \lambda|} \le \frac{\pi}{\gamma}|k|^\tau \qquad \forall k \in \mathbb{Z}\setminus\{0\}, \tag{15}$$

LEMMA 3.6. *Let τ, δ be positive real numbers. Then the series $\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^\tau e^{-|k|\delta}$ is convergent and there exists a constant $C_\tau^{[3.6]}$ depending on τ such that*

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^\tau e^{-|k|\delta} \le C_\tau^{[3.6]}\delta^{-\tau-1} \tag{16}$$

The affine equation (14) is solved by reducing the linear part p to constant coefficients. Next lemma contains the estimates for such reduction.

LEMMA 3.7. *If $p \in \mathscr{P}_\rho$, then for all $\delta > 0$ there exists a function $c \in \mathscr{H}_{\rho-\delta}$ such that*

$$p(z)c(z) = \lambda\, c(z+\omega)$$

for all $z \in K_{\rho-\delta}$ where $\lambda = \exp(\frac{1}{2\pi}\int_0^{2\pi}\log(p(z))\,dz)$. Moreover, there exists a constant $C_\omega^{[3.7]}$ such that

$$\max\left\{\|c\|_{\rho-\delta}, \left\|\frac{1}{c}\right\|_{\rho-\delta}\right\} \le \exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho\delta^{-\tau-1}).$$

Proof. Let us consider the principal value of the logarithm defined on $\mathbb{C}\setminus]-\infty,0]$. The function $\log\circ p$ is analytic on K_ρ, because of the condition $p(K_\rho)$ is a subset of $\mathbb{C}\setminus]-\infty,0]$. Since $\log\circ p \in \mathscr{H}_\rho$, there exists a sequence $\{p_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ such that for all $z \in K_\rho$,

$$\log(p(z)) = \sum_{k\in\mathbb{Z}} p_k e^{ikz}, \qquad \text{with} \quad p_k := \frac{1}{2\pi}\int_0^{2\pi}\log(p(z))e^{-ikz}\,dz.$$

Note that $p_0 = \log\lambda$. This sequence has exponential decay according to Lemma 3.3, $|p_k| \le e^{-k\rho}\|\log\circ p\|_\rho$. Expanding in Fourier series, it is easy to check that the equation $\log(p(z)) + d(z) = p_0 + d(z+\omega)$ has the following formal solution

$$d(z) := d_0 + \sum_{k\in\mathbb{Z}\backslash\{0\}} \frac{p_k}{e^{ik\omega}-1} e^{ikz},$$

for any $d_0 \in \mathbb{C}$. For the sake of simplicity we choose $d_0 = 0$. Let us see that this series defines an analytic function on a suitable domain. Using Lemma 3.5 and the decay of the analytic functions on K_ρ provided by Lemma 3.3 we have that, for all $z \in K_{\rho-\delta}$,

$$\left| \frac{p_k}{e^{ik\omega}-1} e^{ikz} \right| \leq \frac{\pi}{\gamma} |k|^\tau \| \log \circ p\|_\rho e^{-|k|\rho} e^{|k|(\rho-\delta)} = \frac{\pi}{\gamma} \|\log \circ p\|_\rho |k|^\tau e^{-|k|\delta}.$$

Since the series $\sum_{k\in\mathbb{Z}\backslash\{0\}} |k|^\tau e^{-|k|\delta}$ is convergent for any $\delta > 0$, applying Lemma 3.4, d is an analytic function on $\mathring{K}_\rho$, and the bound provided by Lemma 3.6,

$$\begin{aligned} \|d\|_{\rho-\delta} &\leq \frac{\pi}{\gamma} \|\log \circ p\|_\rho \sum_{k\in\mathbb{Z}\backslash\{0\}} |k|^\tau e^{-|k|\delta} \leq \frac{\pi}{\gamma} \|\log \circ p\|_\rho C_\tau^{[3.6]} \delta^{-\tau-1} \\ &= C_\omega^{[3.7]} \delta^{-\tau-1} \|\log \circ p\|_\rho, \end{aligned}$$

being $C_\omega^{[3.7]} = \frac{\pi}{\gamma} C_\tau^{[3.6]}$. So, $c(z) = e^{d(z)}$ is an analytic function on $\mathring{K}_\rho$ such that $e^{p_0} c(z+\omega) = p(z)c(z)$ and, moreover,

$$|c(z)| \leq e^{|d(z)|} \leq \exp(C_\omega^{[3.7]} \|\log \circ p\|_\rho \, \delta^{-\tau-1}), \qquad \forall z \in K_{\rho-\delta}.$$

As $1/c(z) = e^{-d(z)}$, we can bound the norm of $1/c$ with the same estimates as c. $\square$

The following lemma is to find the invariant curve of an affine system after its linear part has been reduced to a constant

LEMMA 3.8. *Let ρ and λ be positive numbers and ω a Diophantine number, see* (11). *If $q \in \mathscr{H}_\rho$ such that $\int_0^{2\pi} q(z)\, dz = 0$ then, for all $\delta > 0$ with $0 < \delta < \rho$, there exists a unique $x \in \mathscr{H}_{\rho-\delta}$ such that $\int_0^{2\pi} x(z)\, dz = 0$ and*

$$x(z+\omega) = \lambda\, x(z) + q(z), \qquad \forall z \in K_{\rho-\delta} \tag{17}$$

Moreover, there exists a constant $C_\omega^{[3.7]}$ such that

$$\|x\|_{\rho-\delta} \leq C_\omega^{[3.7]} \delta^{-\tau-1} \|q\|_\rho.$$

If $q \in \mathscr{S}_\rho^-$, then $x \in \mathscr{S}_{\rho-\delta}^-$.

Proof. There exists a sequence $\{q_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ such that $q(z) = \displaystyle\sum_{k\in\mathbb{Z}\backslash\{0\}} q_k e^{ikz}$ for all $z \in K_\rho$ with $q_k := \frac{1}{2\pi}\int_0^{2\pi} q(z) e^{-ikz}\, dz$. Let us define formally the solution of (17)

$$x(z) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{q_k}{e^{ik\omega}-\lambda} e^{ikz}$$

Because of Lemma 3.5 for a Diophantine number and the exponential decay of analytic functions (Lemma 3.3), it is possible to estimate the term of the series

$$\left|\frac{q_k}{e^{ik\omega}-\lambda} e^{ikz}\right| = \frac{|q_k||e^{ikz}|}{|e^{ik\omega}-\lambda|} \le \frac{\pi}{\gamma}|k|^\tau \|q\|_\rho e^{-\rho|k|} e^{|k|(\rho-\delta)} = \frac{\pi}{\gamma}\|q\|_\rho |k|^\tau e^{-|k|\delta},$$

for $z \in K_{\rho-\delta}$. The series $\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^\tau e^{-|k|\delta}$ is convergent. By Lemma 3.4, the series defining $x(z)$ converges on $\mathring{K}_\rho$ and the following bound is given

$$\|x\|_{\rho-\delta} \le \frac{\pi}{\gamma}\|q\|_\rho \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^\tau e^{-|k|\delta} \le \frac{\pi}{\gamma}\|q\|_\rho C_\tau^{[3.6]}\delta^{-\tau-1} = C_\omega^{[3.7]}\delta^{-\tau-1}\|q\|_\rho,$$

being $C_\omega^{[3.7]} = C_\tau^{[3.6]}\pi/\gamma$, we have chosen $C_\omega^{[3.7]}$ instead of $C_\omega^{[3.8]}$, since it is the same constant of the previous lemma. Indeed, the series is a solution of the equation

$$x(z+\omega) - \lambda x(z) = \sum_{z\in\mathbb{Z}} \frac{q_k e^{ik\omega}}{e^{ik\omega}-\lambda} e^{ikz} - \sum_{k\in\mathbb{Z}} \frac{\lambda q_k}{e^{ik\omega}-\lambda} e^{ikz} + \sum_{k\in\mathbb{Z}} q_k e^{ikz} = \lambda x(z) + q(z)$$

for all $z \in \mathring{K}_\rho$. Moreover, for $q \in \mathscr{S}_\rho^-$ the term with even indices are null. So the series $x(z)$ either and $x \in \mathscr{S}_{\rho-\delta}^-$ for all $\delta \in (0,\rho)$.

Uniqueness: We assume x_1 and x_2 are two solutions for (17). The difference between them is $y(z) := x_1(z) - x_2(z)$ verifies the equation $y(z+\omega) = \lambda g(z)$ for all $z \in K_{\rho-\delta}$ and $\int_0^{2\pi} y(z)\,dz = 0$. Since, y is continuous and $\int_0^{2\pi} y(z)\,dz = 0$, there exists a $z_0 \in [0,2\pi]$ such that $y(z_0) = 0$. With the equation $y(z+\omega) = \lambda y(z)$ and the periodicity of y, we get $y(z + k\omega + j\,2\pi) = 0$. As $z_0 + \mathbb{Z}\omega + 2\pi\mathbb{Z}$ is a dense set in $\mathbb{R}$, we conclude that $y(z) = 0$ for all $z \in K_{\rho-\delta}$. $\square$

Lemma 3.9. *Let ρ be a positive number, $\omega \in \mathbb{R}$ such that verifies (11) $p \in \mathscr{P}_\rho \cap \mathscr{S}_\rho^+$, $q \in \mathscr{S}_\rho^-$, then for all $\delta > 0$ there exists an unique function $x \in \mathscr{S}_{\rho-\delta}^-$ such that*

$$x(z+\omega) = p(z)\,x(z) + q(z), \qquad \forall z \in K_{\rho-\delta}. \tag{18}$$

The solution of (18) *is denoted by $\eta(p)q$ in the following sections. Moreover, there exists a constant $C_\omega^{[3.9]}$ fulfilling the following estimate:*

$$\|\eta(p)q\|_{\rho-\delta} \le \exp(C_\omega^{[3.9]}(1 + \|\log\circ p\|_\rho)\delta^{-\tau-1})\|q\|_\rho. \tag{19}$$

Proof. The number $\tilde\omega = 2\omega$ verifies $|k(2\omega) - 2\pi m| \ge \dfrac{\gamma}{|2k|^\tau} = \dfrac{\tilde\gamma}{|k|^\tau}$ with $\tilde\gamma = \gamma 2^{-\tau}$ and $C_{\tilde\omega}^{[3.5]} = C_\omega^{[3.5]}2^\tau$ and, analogously, $C_{\tilde\omega}^{[3.7]} = C_\omega^{[3.7]}2^\tau$.

The function $z \mapsto p(z/2)$ is analytic on $K_{2\rho}$ and it does not take negative or zero values. So, that function belongs to $\mathscr{P}_{2\rho}$ and by Lemma 3.7 for $\delta > 0$, there exist

$c \in \mathscr{H}_{2\rho-\delta}$ and $\lambda > 0$ such that $\lambda c(z+2\omega) = p(z/2)c(z)$ for all $z \in K_{2\rho-\delta}$ and

$$\max\{\|c\|_{2\rho-\delta}, \|\frac{1}{c}\|_{2\rho-\delta}\} \leq \exp(C_{\tilde{\omega}}^{[3.7]}\|\log\circ p\|_\rho \delta^{-\tau-1}) = \exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho 2^\tau \delta^{-\tau-1}).$$

We define the function $d \in \mathscr{S}^-_{\rho-\delta/2}$ by the quotient

$$d(z) := \frac{q(z)}{c(2(z+\omega))}, \qquad \forall z \in K_{\rho-\delta/2}.$$

Indeed, d is analytic since c is not vanished on $K_{2\rho-\delta}$, 2π–periodic and

$$d(z+\pi) = \frac{q(z+\pi)}{c(2(z+\pi+\omega))} = \frac{-q(z)}{c(2(z+\omega))} = -d(z), \qquad \forall z \in K_{\rho-\delta/2}.$$

By Lemma 3.8 there exists a function $y \in \mathscr{S}^-_{\rho-\delta}$ such that $y(z+\omega) = \lambda y(z) + d(z)$ and

$$\|y\|_{\rho-\delta} \leq C_\omega^{[3.7]}(\frac{\delta}{2})^{-\tau-1}\|d\|_{\rho-\delta/2} \leq C_\omega^{[3.7]}2^{\tau+1}\delta^{-\tau-1}\exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho 2^\tau\delta^{-\tau-1})\|q\|_\rho$$

The function $x(z) := c(2z)y(z)$ belongs to $\mathscr{H}_{\rho-\delta}$ and verifies

$$\begin{aligned} x(z+\pi) &= c(2(z+\pi))y(z+\pi) = c(z)(-y(z)) = -x(z), \\ x(z+\omega) &= c(2(z+\omega))y(z+\omega) = \lambda c(2z+2\omega)y(z) + c(2z+2\omega)d(z) = \\ &= p(2z/2)c(2z)y(z) + q(z) = p(z)x(z) + q(z), \end{aligned}$$

and the estimate for all $z \in K_{\rho-\delta}$

$$\begin{aligned} |x(z)| &\leq \left(\max_{z\in K_{2\rho-\delta}}|c(z)|\right)|y(z)| \\ &\leq \exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho 2^\tau\delta^{-\tau-1})C_\omega^{[3.7]}2^{\tau+1}\delta^{-\tau-1}\exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho 2^\tau\delta^{-\tau-1})\|q\|_\rho \\ &\leq C_\omega^{[3.7]}2^{\tau+1}\delta^{-\tau-1}\exp(C_\omega^{[3.7]}\|\log\circ p\|_\rho 2^{\tau+1}\delta^{-\tau-1})\|q\|_\rho. \end{aligned}$$

taking $C_\omega^{[3.9]} = C_\omega^{[3.7]}2^{\tau+1}$ and using that $t \leq e^t$ for t positive real numbers, the estimation (19) is proved.

Uniqueness: Analogous to the uniqueness in the Lemma 3.8. □

Lemma 3.9 means that there exists a functional $\eta : \mathscr{P}_\rho \to \mathscr{L}(\mathscr{S}^-_\rho, \mathscr{S}^-_{\rho-\delta})$ such that $\eta(p)q$ with $p \in \mathscr{P}_\rho$ and $q \in \mathscr{S}^-_{\rho-\delta}$ is the only function such that $T_\omega(\eta(p)q) = p\cdot(\eta(p)q)+q$. Checking the linearity of the function $\eta(p)$ is trivial and the continuity is got using Cauchy's inequality.

3.2. *Conditions for the convergence of Newton method*

The key of the main theorem in this paper is to find a solution of the equation of the invariant curve by Newton method scheme, once a good enough approximation to the solution is provided. The difficult point is to prove is that the quadratic convergence of the Newton method overcomes the slight loss of the exponential decay of the Fourier coefficients produced for each iteration. These techniques are analogous to the conventional KAM theory. However, an adaptation to the framework of this paper is needed, being that the aim of this section.

The proof of the following lemma can be found in the Appendix.

LEMMA 3.10. *Let U an open set on $\mathbb{C}$ and ϕ an analytic function on U. If x_1 and x_0 belong to the set $\mathcal{U}_\rho := \{x \in \mathscr{H}_\rho \,:\, x(z) \in U \quad \forall x \in K_\rho\}$ then,*

$$\|\phi \circ x_1 - \phi \circ x_0 - (\phi' \circ x_0) \cdot (x_1 - x_0)\|_\rho \leq \frac{1}{2}\|\phi''\|_{\mathscr{C}^0(\overline{U})}\|x_1 - x_0\|_\rho^2. \tag{20}$$

The function whose zeros are invariant curves of is $F(x) = T_\omega x - \phi \circ x - \psi$. Taking the derivative in an adequate functional space, we obtain $DF(x)h = T_\omega h - (\phi' \circ x)h$. The Newton method scheme is given by the equation $DF(x_n)(x_{n+1} - x_n) = -F(x_n)$. Expanding both terms, an equation of the type studied by Lemma 3.9 is obtained:

$$T_\omega x_{n+1} - (\phi' \circ x_n)x_{n+1} = T_\omega x_n - (\phi' \circ x_n)x_n - T_\omega x_n + \phi \circ x_n + \phi, \tag{21}$$

being ϕ and ψ such that Lemma 3.9 can be applied.

Proof of Theorem 3.1. The exponential decay for every iteration decreases a small amount, δ_k. The sum of all of the partial loss of exponential decays must be the overall decreasing in the exponential decay, δ. The analyticity width in the k iteration is denoted by ρ_k,

$$\delta_k := \frac{\delta}{(\frac{\pi^2}{6} - 1)(k+1)^2} \qquad \text{and} \quad \rho_k := \rho_{k-1} - \delta_k \qquad \text{with } k \geq 1.$$

Indeed, using that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, we have that $\displaystyle\sum_{k=1}^{\infty} \delta_k = \frac{\delta}{\frac{\pi^2}{6} - 1} \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \delta$. A exponential decay converges to zero faster that δ_k, hence the series $\delta_k^{-\tau-1}2^{-k-1}$ is convergent

$$\sum_{k=1}^{\infty} \delta_k^{-\tau-1} 2^{-k+1} = (\pi^2/6 - 1)^{\tau+1} \delta^{-\tau-1} \sum_{k=1}^{\infty} (k+1)^{2(\tau+1)} 2^{-k+1} = S\delta^{-\tau-1},$$

where $S := (\pi^2/6 - 1)^{\tau+1} \displaystyle\sum_{k=1}^{+\infty} (k+1)^{2(\tau+1)} 2^{-k+1}$.

For simplicity in the notation we define $\alpha = C_\omega^{[3.9]}(1 + \|\log \circ \phi'\|_{\mathscr{C}^0(\overline{U})})$ and $C_\omega^{[3.1]} = C_\omega^{[3.9]} S$. With this notation the condition (13) becomes $\|T_\omega x_0 - \phi \circ x_0 - \psi\|_\rho < \frac{d}{1 + \|\phi''\|} \exp(-\delta^{-\tau-1} S\alpha)$.

We define recursively the following sequence x_n which are the solutions of (21) for all the iterations

$$x_{n+1} = x_n - \eta(\phi' \circ x_n)(T_\omega x_n - \phi \circ x_n - \psi). \tag{22}$$

In order to check that $x_n \in \mathscr{H}_{\rho_n}$ is well-defined, we are going to prove by induction for $n \geq 0$ that

$$
\begin{aligned}
&(\text{H-1})_n \quad x_n(K_{\rho_n}) \subseteq U \\
&(\text{H-2})_n \quad \|x_{n+1} - x_n\|_{\rho_{n+1}} \leq \frac{d^{2^n}}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp\left(-\alpha 2^n \sum_{k=n+2}^{\infty} \delta_k^{-\tau-1} 2^{1-k}\right)
\end{aligned}
$$

The condition $(\text{H-1})_0$ is one of the hypothesis of the lemma and $(\text{H-2})_0$ is proved by means of Lemma 3.9 and condition (13)

$$
\begin{aligned}
\|x_1 - x_0\|_{\rho_1} &= \|\eta(\phi' \circ x_0)(T_\omega x_0 - \phi \circ x_0 - \psi)\|_{\rho_1} \leq \\
&\leq \exp(\delta_1^{-\tau-1} C_\omega^{[3.9]} (1 + \|\log \circ \phi' \circ x_0\|_\rho)) \|T_\omega x_0 - \phi \circ x_0 - \psi\|_\rho \leq \\
&\leq \exp(\delta_1^{-\tau-1}\alpha) \frac{d}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp(-\delta^{-\tau-1} S\alpha) \leq \\
&\leq \exp(\delta_1^{-\tau-1}\alpha) \frac{d}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp(-\left(\sum_{k=1}^{\infty} \delta_k^{-\tau-1} 2^{-k+1}\right)\alpha) \leq \\
&\leq \frac{d}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp(-\left(\sum_{k=2}^{\infty} \delta_k^{-\tau-1} 2^{-k+1}\right)\alpha)
\end{aligned}
$$

Obviously, we have also the estimate $\|x_1 - x_0\|_{\rho_1} \leq d$.

We assume that $(\text{H-1})_n$ and $(\text{H-2})_n$. Because of $(\text{H-2})_n$, we have that $x_{n+1} \in \mathscr{H}_{\rho_{n+1}}$ with the definition given for the sequence δ_n. Now, we prove that $(\text{H-1})_n$ and $(\text{H-2})_n$ imply $(\text{H-1})_{n+1}$.

From the hypothesis $(\text{H-2})_k$ with $k \leq n$ is hold that $\|x_{k+1} - x_k\|_{\rho_{n+1}} \leq d^{2^k}$ for $k \leq n$. With this, we have the estimate for all $z \in K_{\rho_{n+1}}$

$$
\begin{aligned}
|\text{Re}(x_{n+1}(z))| &\leq \sum_{j=0}^{n} |\text{Re}(x_{n+1-j}(z)) - \text{Re}(x_{n-j}(z))| + |\text{Re}(x_0(z))| \leq \\
&\leq \sum_{j=0}^{n} \|x_{n+1-j} - x_{n-j}\|_{\rho_{n+1}} + R - 2d \\
&\leq \sum_{j=0}^{n} d^{2^{n-j}} + R - 2d \leq \sum_{j=1}^{\infty} d^j + R - 2d \leq d \sum_{j=0}^{\infty} (\frac{1}{2})^j + R - 2d \leq R.
\end{aligned}
$$

The imaginary part has a similar estimation and it is proved in analogous way.

Since $\phi'(\overline{U}) \subseteq \mathbb{C} \setminus (-\infty, 0]$ and ϕ' is an even function, Lemma 3.9 implies that $\phi' \circ x_{n+1}$ belongs to $\mathscr{P}_\rho \cap \mathscr{S}_\rho^+$ and we have that

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\|_{\rho_{n+2}} &= \|\eta(\phi' \circ x_{n+1})(T_\omega x_{n+1} - \phi \circ x_{n+1} - \psi)\|_{\rho_{n+2}} \qquad (23) \\
&\leq \exp(C_\omega^{[3.9]}(1 + \|\log \circ \phi' \circ x_{n+1}\|_{\rho_{n+1}}) \delta_{n+2}^{-\tau-1}) \|T_\omega x_{n+1} - \phi \circ x_{n+1} - \psi\|_{\rho_{n+1}} \\
&\leq \exp(\alpha \delta_{n+2}^{-\tau-1}) \|T_\omega x_{n+1} - \phi \circ x_{n+1} - \psi\|_{\rho_{n+1}}.
\end{aligned}
$$

We have the following relation, applying the shift operator to the definition of the sequence (22),

$$\begin{aligned} T_\omega x_{n+1} &= T_\omega x_n - T_\omega \eta(\phi' \circ x_n)(T_\omega x_n - \phi \circ x_n - \psi) = \\ &= T_\omega x_n - (\phi' \circ x_n) \cdot \eta(\phi' \circ x_n)(T_\omega x_n - \phi \circ x_n - \psi) - T_\omega x_n + \phi \circ x_n + \psi = \\ &= (\phi' \circ x_n) \cdot (x_{n+1} - x_n) + \phi \circ x_n + \psi. \end{aligned}$$

So, $T_\omega x_{n+1} - \phi \circ x_{n+1} - \psi = -\phi \circ x_{n+1} + \phi \circ x_n + (\phi' \circ x_n) \cdot (x_{n+1} - x_n)$ and, returning to the estimation (23), we have that

$$\|x_{n+2} - x_{n+1}\|_{\rho_{n+2}} \leq \exp(\alpha \delta_{n+2}^{-\tau-1}) \|\phi \circ x_{n+1} - \phi \circ x_n - (\phi' \circ x_{n+1}) \cdot (x_{n+1} - x_n)\|_{\rho_{n+1}}.$$

Now, Lemma 3.10 implies that

$$\|x_{n+2} - x_{n+1}\|_{\rho_{n+1}} \leq \exp(\alpha \delta_{n+2}^{-\tau-1}) \frac{1}{2} \|\phi''\|_{\mathscr{C}^0(\overline{U})} \|x_{n+1} - x_n\|^2_{\rho_{n+1}}.$$

and, by the induction hypothesis (H-1)$_n$, we conclude that

$$\begin{aligned} \|x_{n+2} - x_{n+1}\|_{\rho_{n+1}} &\leq \exp(\alpha \delta_{n+2}^{-\tau-1}) \frac{1}{2} \|\phi''\|_{\mathscr{C}^0(\overline{U})} \left(\frac{d^{2^n}}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp(-\alpha \sum_{k=n+2}^{\infty} \delta_k^{-\tau-1} 2^{n+1-k}) \right)^2 \\ &\leq \frac{d^{2^{n+1}}}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}} \exp(-\alpha \sum_{k=n+3}^{\infty} \delta_k^{-\tau-1} 2^{n+2-k}) \end{aligned}$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathscr{C}^0(K_{\rho_0-\delta})$, since

$$\|x_n - x_m\|_{\rho_0-\delta} \leq \sum_{k=m+1}^{n} \|x_k - x_{k-1}\|_{\rho_k} < \sum_{k=m}^{n} (\frac{1}{2})^{2^{k-1}} \tag{24}$$

and the series $\sum_{k=0}^\infty (\frac{1}{2})^{2^{k-1}}$ converges. Therefore, x_n converges to a continuous function x on the compact $K_{\rho_0-\delta} \cap ([-j\pi, j\pi] + i\mathbb{R})$ for any $j \in \mathbb{N}$. Taking limits on (24) $\|x_n - x\|_{\rho_0-\delta} < \sum_{k=n}^\infty (\frac{1}{2})^{2^{k-1}}$ and x_n converges to x for any compact on $\mathring{K}_\rho$. Therefore, x is analytic on $\mathring{K}_\rho$.

The function $x \in \mathscr{S}^-_{\rho-\delta}$ because it is the limit of a sequence in a closed set. The equation $T_\omega x = \phi \circ x + \psi$ is fulfilled due to be the solution of the Newton method scheme. $\square$

Proof of Corollary 3.2. We can assume $a_0 = 1$. Obviously, the set $x_0(\mathbb{R}) = x_0([0, 2\pi])$ is compact, and there exists $A > 0$ such that $x_0([0, 2\pi])$ is a subset of $[-A, A]$. Since, the function f is analytic in $[-A, A]$, we can extend the defined function f to a set $U_1 = [-A - \epsilon, A + \epsilon] + i[-\epsilon, \epsilon]$ where the function f is analytic and the real part of the derivative is nonnegative, i.e. $\mathrm{Re}(f'(z)) \geq 0$ for $z \in U_1$. Moreover, there exists a $M > 0$ such that the function and the two first derivatives are bounded by M, i.e. $|f(z)| \leq M$, $|f'(z)| \leq M$ and $|f''(z)| \leq M$ for all $z \in U_1$.

There exists $\rho > 0$ and $\epsilon > 0$ such that the function x_0 and g are extended to an analytic function and the set $x_0(K_{2\rho})$ is included in the set

$$U = [-A - \epsilon/2, A + \epsilon/2] + i[-\epsilon/2, \epsilon/2]$$

and there is a positive number $L > 0$ such that $|g(z)| < L$ for all $z \in K_{2\rho}$. Let us define a constant depending on the function f via M and ϵ, the shift ω and the band where the initial solution is analytical, ρ

$$C := \frac{\epsilon}{4} \frac{\exp(-(\rho/2)^{-\tau-1} C_{\omega}^{[3.1]}(\log(4M/3) + 2\pi + 1)}{1 + 16M/9}$$

The function $z \in K_{2\rho} \mapsto x_0(z+\omega) - f(x_0(z)) - b_0 g(z)$ is analytic and is vanished along real axis. Therefore, $x_0(z+\omega) = f(x_0(z)) + b_0 g(z)$ for all $z \in K_{2\rho}$. Analogously, for the other conditions $x_0(z+\pi) = -x_0(z)$ and $g(z+\pi) = -g(z)$ for all $z \in K_{2\rho}$.

If $|a-1| \leq \max(\frac{C}{2M\|x_0\|_\rho}, 1/3)$ and $|b - b_0| \leq \min(\frac{C}{2L}, 1/2)$, then the range of $a\,x_0$ is included in U_1 where f is defined, since the absolute value of the imaginary part of $a\,x_0(z)$ is less than $2\epsilon/3$. On the other hand,

$$\begin{aligned} |x_0(z+\omega) - f(a\,x_0(z)) - b\,g(z)| &= |f(x_0(z)) + b_0 g(z) - f(a\,x_0(z)) - b\,g(z)| \\ &\leq |f(x_0(z)) - a\,f(x_0(z))| + |(b - b_0)g(z)| \\ &\leq |1 - a| M \|x_0\|_\rho + |b - b_0| L \leq C \end{aligned}$$

is verified for all $z \in K_\rho$. We apply Lemma 3.1 with $\phi(z) = f(a\,z)$ and $\psi(z) = b\,g(z)$.

$$\|\log \circ \phi'\|_{\overline{U}} \leq \log(|a|\,M) + 2\pi \leq \log(4M/3) + 2\pi$$

and $1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})} \leq 1 + |a|^2 M \leq 1 + 16M/9$. Then

$$C \leq \frac{\epsilon}{4} \frac{\exp(-(\rho/2)^{-\tau-1} C_{\omega}^{[3.1]}(\|\log \circ \phi'\|_{\overline{U}} + 1)}{1 + \|\phi''\|_{\mathscr{C}^0(\overline{U})}}$$

so, there is $x \in \mathscr{S}_\rho^-$ such that $x(z+\omega) = f(a\,x(z)) + b\,g(z)$. □

4. Conclusions

One of the main contributions of this note is to depict numerical grounds which point out the appearance of non-analytical invariant curves, reaffirming the conjecture of the existence of SNAs for this system and establishing the boundaries of the region where these curves might dwell. Nevertheless, a rigorous proof is still required and we hope that this work will provide different perspectives on this question.

On the other hand, in this context the introduction of reduction techniques for continuation constitutes a tool which makes possible thorough studies of invariant curves in forced 1-D maps. Lastly, the persistence of properties, e.g. analyticity, for symmetric curves is another path to penetrate in the knowledge of these maps.

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Supplemental material

The git project `https://github.com/fjmalmaraz/SymRedInvCurves` has been created with the code for symmetric invariant curves calculation.

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Appendix A. Proofs of technical lemmas

Proof of Lemma 3.4. The series $\sum f_k$ is uniformly convergent on the compact $K_{\rho-\delta} \cap ([-j\pi, j)\pi] + i\mathbb{R})$ for any $j \in \mathbb{Z}$, since f_k is bounded by a a convergent series on that compact. Being $\delta > 0$ arbitrary, the functional series converges compactly on $\mathring{K}_\rho$. Therefore, $f(z)$ is an analytic function on the interior of K_ρ and the given bound. $\square$

Proof of Lemma 3.5. Considering angles with the minimum absolute value, i.e. the interval $(-\pi, \pi)$. As $|\sin\theta| \geq 2|\theta|/\pi > |\theta|/\pi$ if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $|\lambda - e^{i\theta}| \geq 1 \geq |\theta|/\pi$

if $\theta \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$, we have that

$$|e^{i\theta} - \lambda| \geq \frac{1}{\pi} \min_{m\in\mathbb{Z}} |\theta - 2m\pi|, \qquad \forall \theta \in \mathbb{R}. \tag{A1}$$

Using equation (11), the following estimate follows:

$$|e^{ik\omega} - \lambda| \geq \frac{1}{\pi} \min_{m\in\mathbb{Z}} |k\omega - 2\pi\, m| \geq \frac{1}{\pi} \frac{\gamma}{|k|^\tau}.$$

Consequently, with the last equation and the bound for the power x^τ is concluded

$$\frac{1}{|e^{ik\omega} - \lambda|} \leq \frac{\pi}{\gamma}|k|^\tau \leq \frac{\pi}{\gamma}(\frac{\tau}{e\delta})^\tau e^{|k|\delta}$$

being the constant given in the lemma $C_\omega^{[3.5]} := \frac{\pi}{\gamma}(\frac{\tau}{e})^\tau$. □

Proof of Lemma 3.10. By Taylor's Theorem, we have

$$\begin{aligned} &\phi(x_1(z)) - \phi(x_0(z)) - \phi'(x_0(z))(x_1(z) - x_0(z)) = \\ &\quad \int_0^1 (1-s)\phi''(x_0(z) + s(x_1(z) - x_0(z)))(x_1(z) - x_0(z))^2\, ds. \end{aligned}$$

So, we have the estimate

$$|\phi(x_1(z)) - \phi(x_0(z)) - \phi'(x_0(z))(x_1(z) - x_0(z))| \leq \frac{1}{2}\|\phi''\|_{\mathscr{C}^0(\overline{U})}\|x_1 - x_0\|_\rho^2,$$

for all $z \in K_\rho$ and (20) holds. □

Proof of Lemma 3.6. Let us consider the real function $f(x) := x^\tau e^{-x\delta}$. Its derivate is $f'(x) = e^{-x\delta}x^{\tau-1}(\tau - \delta\, x)$. Obviously, the function reaches the maximum on $x = \tau/\delta$ and $f(\tau/\delta) = (\tau/(\delta e))^\tau$. Hence, $x^\tau \leq \left(\frac{\tau}{e\delta}\right)^\tau e^{x\delta}$ for all $x > 0$. Applying this inequality with $x = k$, the following bound guarantees the convergence of the series

$$\begin{aligned} |k|^\tau e^{-|k|\delta} &\leq \left(\frac{2\tau}{e\delta}\right)^\tau e^{-|k|\delta/2}, \\ \sum_{k\in\mathbb{Z}\backslash\{0\}} |k|^\tau e^{-|k|\delta} &\leq \left(\frac{2\tau}{e\delta}\right)^\tau \sum_{k\in\mathbb{Z}\backslash\{0\}} e^{-|k|\delta/2} \leq \left(\frac{2\tau}{e\delta}\right)^\tau \frac{2}{e^{\delta/2} - 1}. \end{aligned}$$

Using the inequality $t \leq e^t - 1$ for all t positive, we get the estimation

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |k|^\tau e^{-|k|\delta} \leq \left(\frac{2\tau}{e\delta}\right)^\tau 2\frac{2}{\delta} = 2\left(\frac{\tau}{e}\right)^\tau \left(\frac{2}{\delta}\right)^{\tau+1} = C_\tau^{[3.6]}\delta^{-\tau-1},$$

where $C_\tau^{[3.6]} = 2^{\tau+2}\left(\frac{\tau}{e}\right)^\tau$. □