Seminari: RTBP III Libration Points Motion

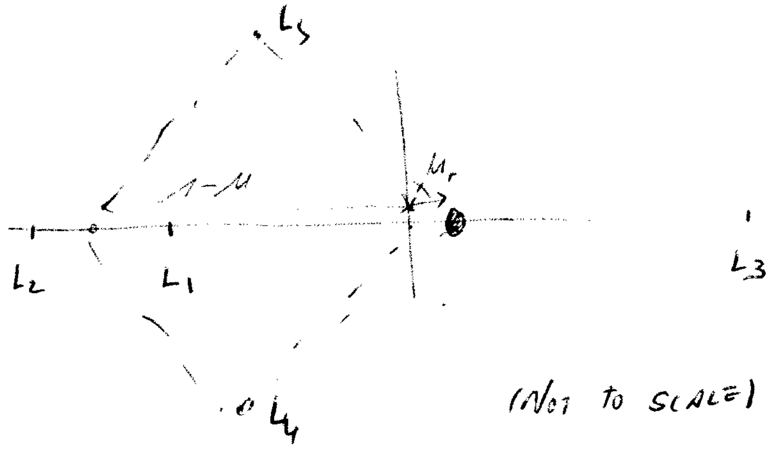
(by A. Farrés)

- Remember eq. motion:

$$\ddot{x} - 2\dot{y} = \Omega_x,$$
$$\ddot{y} + 2\dot{x} = \Omega_y, \quad \text{on} \quad \Omega(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$
$$\ddot{z} = \Omega_z.$$

- Eq. Points:

  · there are five: L_i



in this talk we will focus on the motion around the collinear points. (L_1, L_2, L_3)

- L_1 and L_2 have been very studied and are very usefull for mission aplications
- We want to understand the periodic and quasi periodic motion around those points.
- As we saw in RTBP Part II these fixed points are of the type centre × centre × saddle.

Motion around the collinear eq points:

1st/ Change of variables to bring the fixed point to the origin:

$$X = \mp \xi_j\, x + \mu + a_j \qquad \text{on } \xi_j \equiv \text{distance to the closest primari / solution of the Euler quintic}$$

$$Y = \mp \xi_j\, y$$

$$Z = \xi_j\, z$$

$$\begin{cases} a_1 = -1 + \xi_1 \\ a_2 = -1 - \xi_2 \\ a_3 = \xi_3 \end{cases}$$

(Nota: Scaling is ~~...~~ done so that the expansion around the fixed point ~~is~~ ~~a~~ has good numerical properties.)

Using Legendre Polinomials;

$$\frac{1}{\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}} = \frac{1}{D}\sum_{n=0}^{\infty}\left(\frac{\rho}{D}\right)^n P_n\left(\frac{Ax+By+Cz}{D\rho}\right),$$

on $D^2 = A^2+B^2+C^2$, $\rho^2 = x^2+y^2+z^2$, P_n ~~polynomial~~ polinomi legendr of degree n.

Eq Expand:

$$\ddot{x} - 2\dot{y} - (1+2c_2)\,x = \frac{\partial}{\partial x}\sum_{n\geq 3} c_n(\mu)\,\rho^n P_n\left(\frac{x}{\rho}\right)$$

$$\ddot{y} + 2\dot{x} + (c_2+1)\,y = \frac{\partial}{\partial y}\sum_{n\geq 3} c_n(\mu)\,\rho^n P_n\left(\frac{x}{\rho}\right)$$

$$\ddot{z} + c_2 z = \frac{\partial}{\partial z}\sum_{n\geq 3} c_n(\mu)\,\rho^n P_n\left(\frac{x}{\rho}\right)$$

or

$$C_n(\mu) = \frac{1}{\xi_j^3}\left((\pm 1)^n \mu + (-1)^n \frac{(1-\mu)\,\xi_j^{n+1}}{(1\mp\xi_j)^{n+1}}\right) \quad \text{for } L_{1,2} \quad j=1,2$$

$$C_n(\mu) = \frac{(-1)^n}{\xi_3^3}\left((1-\mu) + \mu \cdot \frac{\xi_3^{n+1}}{1+\xi_3}\right) \quad \text{for } L_3$$

Stability of the fixed Points (vap's & vep's)

$$Df_{(0,0)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1+C_2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1-C_2 & 0 & -2 & 0 & 0 \\ 0 & 0 & -C_2 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Notice that the $\{z, \dot{z}\}$ plane/motion is decoupled from the rest $\{x, y, \dot{x}, \dot{y}\}$

- $p_6(\lambda) = \underbrace{[\lambda^4 + (2-C_2)\lambda^2 + (1+C_2-2C_2^2)]}_{\text{rest}} \cdot \underbrace{(\lambda^2 + C_2)}_{\text{associated to } z, \dot{z}}$

taking $\nu = \lambda^2$ we can see that the roots are:

$$\nu_1 = \frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2} \quad ; \quad \nu_2 = \frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}$$

As $c_2 > 1 \quad \forall \mu \in [0, \frac{1}{2}]$

$\Rightarrow \nu_1 > 0 \; ; \; \nu_2 < 0 \quad \Rightarrow \quad \omega_1 = \sqrt{-\nu_1}$ complex eigenvalue

$\lambda = \sqrt{\nu_2}$ real eigenvalues

$\omega_2 = \sqrt{-c_2}$ complex eigenvalue

$\Rightarrow$ fixed points are ~~unknown~~
centre $\times$ centre $\times$ saddle.

One can easily compute the eigenvectors and see:

$$u_\lambda = (2\lambda, \lambda^2 - 2c_2 - 1, \; 0, \; 2\lambda^2, \; \lambda(\lambda^2 - (1+2c_2)), \; 0)$$

$$u_{-\lambda} = (-2\lambda, \lambda^2 - 2c_2 - 1, \; 0, \; 2\lambda^2, \; -\lambda(\lambda^2 - (1+2c_2)), \; 0)$$

$$u_{\omega_1} = (0, -\omega_1^2 - 1 - 2c_2, \; 0, \; -2\omega_1^2, \; 0, \; 0)$$

$$v_{\omega_1} = (2\omega_1, \; 0, \; 0, \; 0, \; -\omega_1(\omega_1^2 + 1 + 2c_2), \; 0)$$

$$u_{\omega_2} = (0, \; 0, \; 1, \; 0, \; 0, \; 0)$$

$$v_{\omega_2} = (0, \; 0, \; 0, \; 0, \; 0, \; 1)$$

- We have 2 central directions + 1 saddle plane.

- **Lyapunov Centre Theorem:** Assume we have a system
We have a non-degenerate 1st integral and an equilibrium point
with $\pm \omega i, \lambda_3, \ldots, \lambda_n$ by $i\omega \neq 0 \in \mathbb{C}$ pure imaginari
If $\lambda_j / i\omega \notin \mathbb{N}$ $\forall j = 3, \ldots, n$ $\Rightarrow \exists$ one-parameter family
of periodic orbits emanating from the eq. point

Assume that as $\omega_1 / \omega_2 \notin \mathbb{Q}$ $\Rightarrow$ we have two families of periodic orbits; one for each frequenci ω_1 and ω_2

→ ω_1 gives rise to the plan family /
(Lyapunov orbits)

→ ω_2 gives rise to the vertical family
(Vertical Lyap. orbits)

We can use a continuation method to follow this families

(→ NOTE: very unstable: suitable to use Parallel Shooting;
other option Lindstedt Poincaré).

- LINEAR APROXIMATION of MOTION AROUND EQUILIBRIA:

$$X(t) = A e^{\lambda t} \cdot \vec{u}_\lambda + B e^{-\lambda t} \cdot \vec{u}_{-\lambda} +$$

$$+ C \cdot [\cos(w_1 t + \varphi_0)\, \vec{u}_{w_1} + \sin(w_1 t + \varphi_0)\, \vec{u}_{w_1}] +$$

$$+ D\, [\cos(w_2 t + \phi_0)\, \vec{u}_{w_2} + \sin(w_2 t + \phi_0)\, \vec{V}_{w_2}]$$

on A, B, C, D, φ_0, ϕ_0 given by the initial condition at $t = t_0$

→ Taking: $A = B = C = 0$ → Vertical family
$A = B = D = 0$ → Planar family.

- STABILITY AROUND PERIODIC ORBIT:

- T-Periodic orbit ⇒ $\varphi_t(x)$ = flow around trajectory

and $M = D\varphi_T(x_0)$ = monodromi matrix for the periodic orbit X_0.

As the system is Hamiltonian I has a 1^{st} Integral

$\Rightarrow$ Spect $M = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$

- We define the stability parametres $s_j = \lambda_j + \lambda_j^{-1}$ $j = 1, 2$.

$\Rightarrow$ <u>Hyperbolic</u>: if $s_j \in \mathbb{R}$; $|s_j| > 2$ $(\lambda_j \in \mathbb{R} \setminus \{1, -1\})$

<u>Elliptic</u>: if $s_j \in \mathbb{C}$, $|s_j| < 2$ $(\lambda_j = e^{i\rho})$

- We have followed the Planar and Vertical Lyapunov families for L_1 and L_2, for μ:

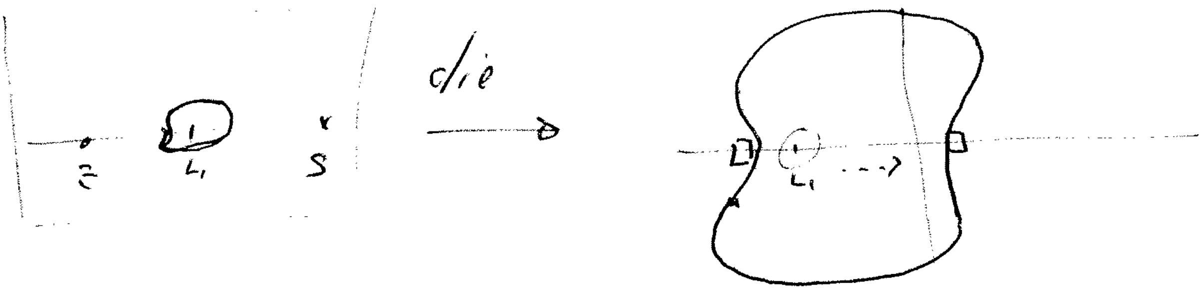
(μ for Sun-Earth model)

- Special attention to 1^{st} bifurcations + where they die.

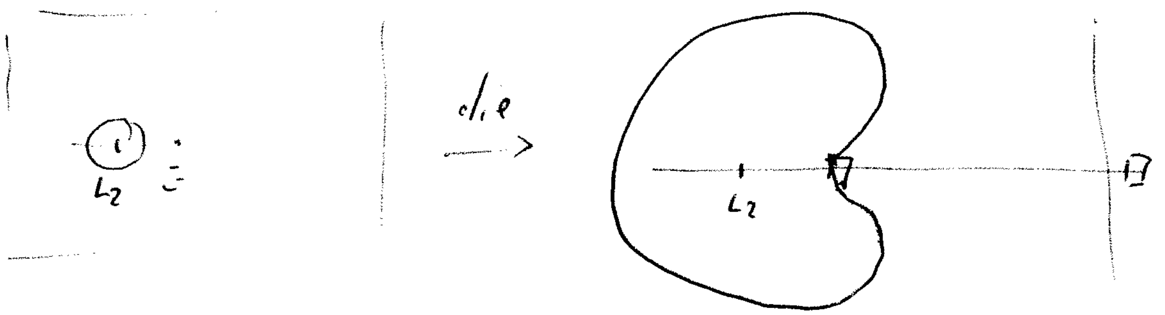
CLASSIFICACIÓ PERIODIC ORBITS.

Planar families:

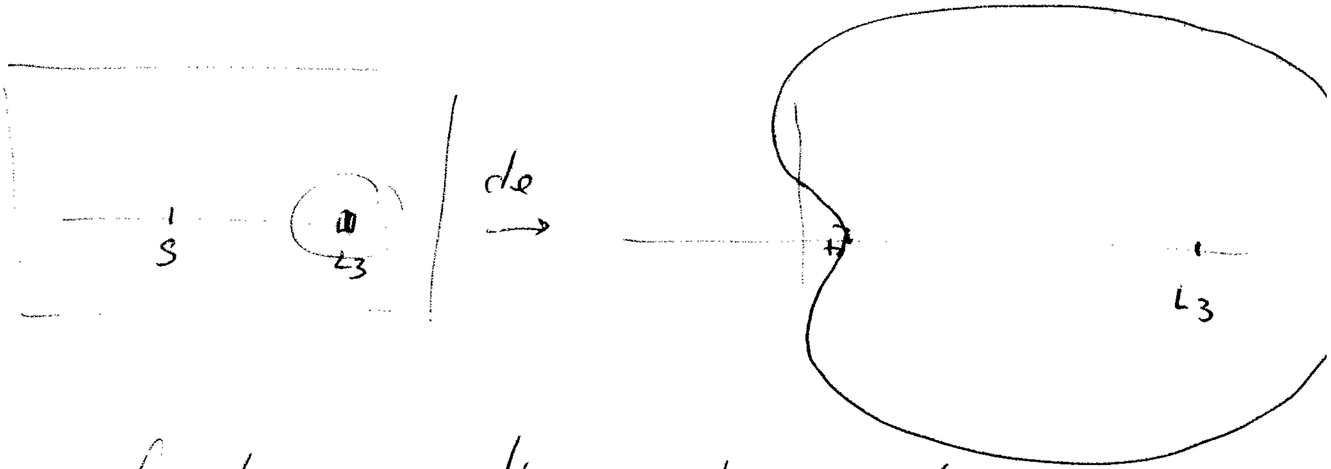
L_1.



L_2



L_3



- L_i planar family dies with a colision with one of the two primaries.
- At the begining L_i planar family bifurcate.

  $C \times S \longrightarrow S \times S$ and the Halo orbits APEAR.

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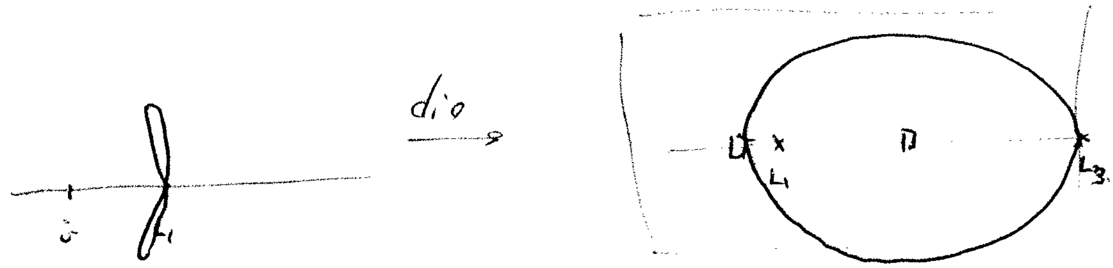
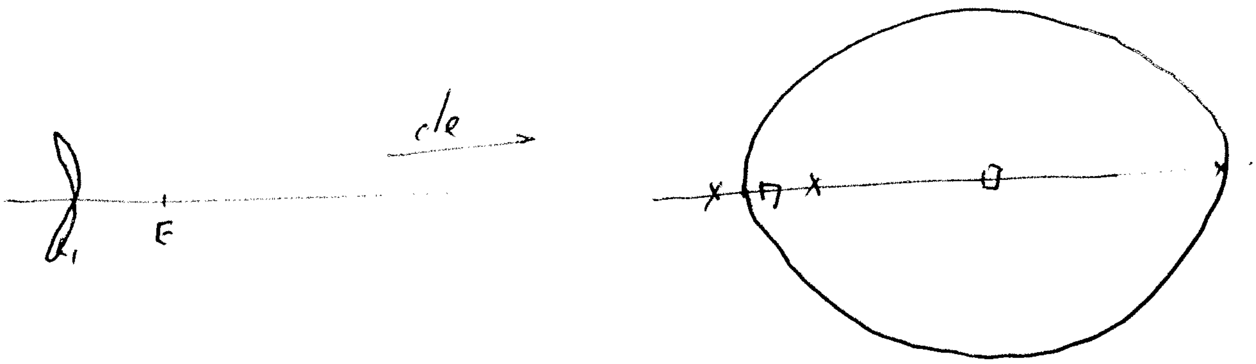
VERTICAL FAMILIES :

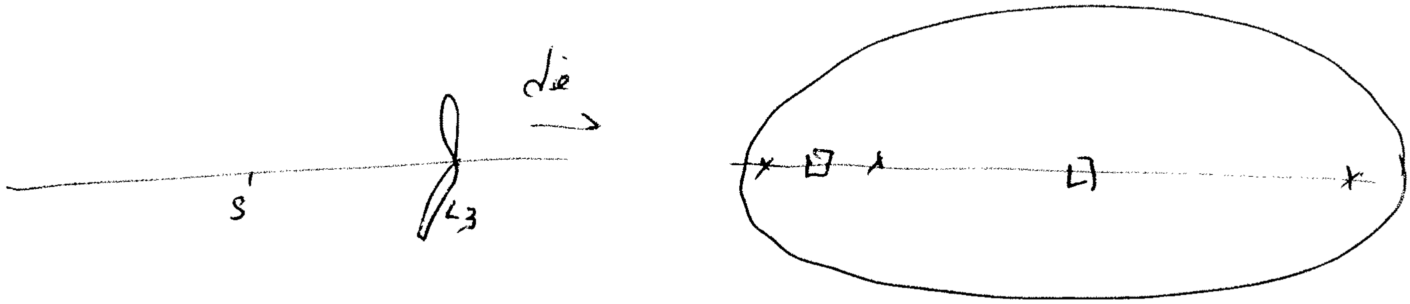
L_1



L_2



L_3



L_1 terminates at: orbit surrounding massive body & L_1 L_3

L_2 " " : orbit surrounding two bodies & L_1 L_3

L_3 " " " " two bodies & L_1 L_2 L_3

• Δ bifurcation appear at the begining

(see procot op_libre.pdf)

Halo families:

- They appear ~~???~~ when PLANAR ORBIT bifurcates.

- Symmetric w.r.t $y = 0$ & $z = 0$

- With this we can see the families of invariant tori that appear around the eq. point

- $H = \lambda x_1 y_1 + i\omega_1 \frac{(x_2^2 + y_2^2)}{2} + i\omega_2 \frac{(x_3^2 + y_3^2)}{2}$

as $\omega_1, \omega_2 > 0 \Rightarrow$ for a fixed $H = h$ energy level the motion is bounded by an ellipsoide on the centre manifold (taking $x_1 = y_1 = 0$).

- Look at pictures of the centre manifold for different h.

→ As h increases $\Rightarrow$ planar Lyap bifurcates and Halo orbits appear.

→ We see all that can be here.

[Plots]

Two dif sections $\underbrace{x_3 = 0}_{z=0}$, $\underbrace{x_4(h)}_{\dot{z}(h)}$; $\underbrace{x_1 = 0}_{y=0}$, $\underbrace{x_2(h)}_{\dot{y}(h) \text{ at } 1^{st} \text{ order}}$

- To have a more complet understanding of the motion it is interesting to do the so-called REDUCTION TO THE CENTRE MANIFOLD.

- Due to the big instability of the eq. points, Poincaré sections & numerical integration gives problems as the trajectories quickly escape from a close vicinity of the eq point.

- By reducing to the centre manifold:
(1) we decouple the elliptic directions from the hyperbolic ones up to high order.
(2) use the high order approximation of the centre manifold to do numerical integrations
(3) How to do this: (see REF)

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