Working seminar on Celestial mechanics
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M. Ollé

Poincaré in his book 'Méthodes nouvelles de la mécanique celeste' defines three types of periodic orbits (PO) in the RTBP for $\mu > 0$ and small:

- First kind: close to keplerian circles
- Second kind: close to keplerian ellipses

both included in the so called 'First species solutions' and

- Second species solutions: close to a set of connected arcs of keplerian ellipses, each such arc is an orbit with consecutive collisions.
- In the 70's a very complete numerical exploration for $0.1 \leq \mu \leq 1/2$ was perfomed by E. Strömgren and his associates in Copenhagen. The classification of the families computed is based on 7 special points of the RTBP: $L_1, ..., L_5$ and the points where the primaries are located.
- Study of invariant manifolds of a collinear point L_i, $i = 1, 2, 3$.

Other applications:

- Space dynamics: using the Earth-Moon system as the primaries in the RTBP, space probe trajectories connecting the two force centers can be established.
- Stellar dynamics: tendency of stars in a cluster to form binaries.
- In general, close approaches, collisions and captures cannot be handled without regularization.

Regularization of the planar RTBP

The equations of motion in synodical coordinates are

$$\left\{ \begin{array}{l} \ddot{x} - 2\dot{y} = \Omega_x \\ \ddot{y} + 2\dot{x} = \Omega_y \end{array} \right. \Longleftrightarrow \ddot{x} + i\ddot{y} + 2i(\dot{x} + i\dot{y}) = \Omega_x + i\Omega_y \Longleftrightarrow \ddot{z} + 2i\dot{z} = \mathrm{grad}_z\Omega \qquad (1)$$

where $\dot{z} = \dfrac{dz}{dt}$, $z = x + iy$ and $\mathrm{grad}_z\Omega = \Omega_x + i\Omega_y$

$$\Omega = \frac{1}{2}\left[(1-\mu)r_0^2 + \mu r_1^2\right] + \frac{1-\mu}{r_0} + \frac{\mu}{r_1}$$
$$r_0 = \sqrt{(x-\mu)^2 + y^2} \qquad r_1 = \sqrt{(x-\mu+1)^2 + y^2}\,.$$

So, the equations become singular when r_0 or $r_1 \to 0$ (collision with either of the primaries).

In order to regularize the eq (1), we consider two transformations

$$\begin{cases} z = f(w) \\ \dfrac{dt}{ds} = g(w) = |f'(w)|^2 \end{cases}$$

with $w = u + iv$.

Then **Proposition**
(i) The transformed eq becomes

$$w'' + 2i|f'|^2 w' = |f'|^2 grad_w \widetilde{\Omega} + \frac{|w'|^2 \overline{f}''}{\overline{f}'}$$

with $\Omega(x, y) = \Omega(x(u, v), y(u, v)) = \widetilde{\Omega}(u, v)$.
(ii) Defining $\mathcal{U} = \widetilde{\Omega} - \dfrac{C}{2}$ *and using the Jacobi integral* ($2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = C$)*, we obtain*

$$w'' + 2i|f'|^2 w' = grad_w(\mathcal{U}|f'|^2) .$$

Proof.

$$\begin{aligned} \dot{z} &= \frac{dz}{dw}\frac{dw}{ds}\frac{ds}{dt} = f' \cdot w' \, \dot{s} \\ \ddot{z} &= f'w'\ddot{s} + f''w'\dot{s}w'\dot{s} + f'w''\dot{s}\,\dot{s} = f'w'\ddot{s} + (f''{w'}^2 + f'w'')\dot{s}^2 \end{aligned}$$

Let us transform $\text{grad}_z\Omega$.

Lemma 1

$$\overline{f}' \text{grad}_z \Omega = \text{grad}_w \widetilde{\Omega}$$

where $\text{grad}_w \widetilde{\Omega} = \widetilde{\Omega}_u + i\widetilde{\Omega}_v$.

Proof.

$$z = x + iy = f(w) = f(u, v) \Longrightarrow \frac{df}{dw} = \frac{\partial f}{\partial u} = x_u + iy_u = -i\frac{\partial f}{\partial v} .$$

From Cauchy-Riemann eqs, $x_u = y_v, \;\; x_v = -y_u$ we have

$$\begin{aligned} \text{grad}_w \widetilde{\Omega} &= \widetilde{\Omega}_u + i\widetilde{\Omega}_v = \Omega_x x_u + \Omega_y y_u + i(\Omega_x x_v + \Omega_y y_v) = \\ &= \Omega_x x_u + \Omega_y y_u + i(-\Omega_x y_u + \Omega_y x_u) = \\ &= (x_u - iy_u)(\Omega_x + i\Omega_y) = \overline{f}' \text{grad}_z \Omega . \qquad \square \end{aligned}$$

So, equation (1) reads

$$f'w'\ddot{s} + (f'w'' + f''w'^2)\dot{s}^2 + 2if'w'\dot{s} = \frac{1}{\overline{f}'}\text{grad}_w \widetilde{\Omega}$$

Dividing by $f'\dot{s}^2$,

$$\frac{w'}{\dot{s}^2}\ddot{s} + w'' + \frac{f''}{f'}w'^2 + 2i\frac{w'}{\dot{s}} = \frac{1}{|f'|^2\dot{s}^2}\text{grad}_w \widetilde{\Omega}$$

and equivalently

$$w'' + w'\frac{\ddot{s}}{\dot{s}^2} + i\frac{2w'}{\dot{s}} = -\frac{w'^2 f''}{f'} + \frac{1}{|f'|^2\dot{s}^2}\text{grad}_w \widetilde{\Omega} . \tag{2}$$

Let us compute $\frac{\ddot{s}}{\dot{s}^2}$:

$$\begin{aligned} \dot{s} &= \frac{1}{g} = \frac{1}{|f'|^2} = \frac{1}{f'\overline{f'}} \qquad (**) \\ \ddot{s} &= -\frac{\dot{g}}{g^2} = -\dot{g}\dot{s}^2 \longleftrightarrow \frac{\ddot{s}}{\dot{s}^2} = -\dot{g} \end{aligned}$$

Lemma 2 $\dot{g} = \frac{\overline{f}''\overline{w}'}{\overline{f}'} + \frac{f''w'}{f'}$.

Proof. $\dot{g} = f'\frac{d\overline{f}'}{dt} + \overline{f}'\frac{df'}{dt} \underset{(*)}{=} \left(f'\overline{f}''\overline{w}' + \overline{f}'f''w'\right)\dot{s} \underset{(**)}{=} \frac{\overline{f}''\overline{w}'}{\overline{f}'} + \frac{f''w'}{f'}$ □

$$(*):\quad \begin{aligned} \frac{df'}{dt} &= \frac{df'}{dw}\cdot\frac{dw}{ds}\dot{s} = f''w'\dot{s} \\ \frac{d\overline{f}'}{dt} &= \frac{\overline{df'}}{dt} = \overline{\frac{df'}{dw}\frac{dw}{ds}}\dot{s} = \overline{f}''\overline{w}'\dot{s} \end{aligned}$$

So, equation (2) becomes

$$w'' - w'\left[\frac{\overline{f}''\overline{w}'}{\overline{f}'} + \frac{f''w'}{w'}\right] + 2i|f'|^2w' = -\frac{w'^2f''}{f'} + |f'|^2\mathrm{grad}_w\widetilde{\Omega}$$

or

$$w'' + 2i|f'|^2w' = \underbrace{|f'|^2\mathrm{grad}_w\widetilde{\Omega} + \frac{|w'|^2\overline{f}''}{\overline{f}'}}_{RT}, \tag{3}$$

as (i) states.

(ii) Now for the right hand side term, we use $U = \widetilde{\Omega} - \frac{C}{2}$, so $\mathrm{grad}_w\widetilde{\Omega} = \mathrm{grad}_w U$.

From the Jacobi integral

$$|\dot{z}|^2 = 2\widetilde{\Omega} - C = 2U \underset{\dot{z}=f'w'\dot{s}}{\Longleftrightarrow} 2U = |f'|^2|w'|^2\dot{s}^2 \underset{\dot{s}^2=\frac{1}{|f'|^4}}{=} \frac{|w'|^2}{|f'|^2} \Longleftrightarrow |w'|^2 = 2|f'|^2U\,.$$

So,

$$RT = |f'|^2\mathrm{grad}_wU + \frac{2|f'|^2U\overline{f}''}{\overline{f}'} = |f'|^2\mathrm{grad}_wU + 2f'\overline{f}''U$$

Now we use

Lemma 3 *If $g_1(w)$, $g_2(w)$, are real analytic functions of a complex variable w, then*

(i) $grad_w(g_1(w)g_2(w)) = g_1 grad_w g_2 + g_2 grad_w g_1$.

(ii) If $G(w)$ is an analytic complex function of a complex variable w, then $grad_w|G(w)|^2 = 2G\overline{\frac{dG}{dw}}$.

Proof.

(i) $\mathrm{grad}_w(g_1g_2) = (g_1g_2)_u + i(g_1g_2)_v =$

$$\begin{aligned} &= g_{1_u}g_2 + g_1g_{2_u} + ig_{1_v}g_2 + ig_1g_{2_v} \\ &= g_1[g_{2_u} + ig_{2_v}] + g_2[g_{1_u} + ig_{1_v}] \\ &= g_1\mathrm{grad}_wg_2 + g_2\mathrm{grad}_wg_1 \end{aligned}$$

(ii) If $G = R + iI$

$$
\begin{aligned}
&|G|^2 = R^2 + I^2 \\
&\mathrm{grad}_w |G|^2 = 2RR_u + 2II_u + i(2RR_v + 2II_v) \\
&2G\frac{d\overline{G}}{dw} = 2(R+iI)(R_u - iI_u) = 2[RR_u + II_u + i(-RI_u + IR_u)] \underset{R_u = I_v,\, R_v = -I_u\ (CR)}{\overset{\uparrow}{=}} \\
&\mathrm{grad}_w |G|^2 . \qquad \square
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{grad}_w(U|f'|^2) &\overset{(i)}{=} U \mathrm{grad}_w |f'|^2 + |f'|^2 \mathrm{grad}_w U \overset{(ii)}{=} \\
&= U 2f'\overline{f}'' + |f'|^2 \mathrm{grad}_w U = RT
\end{aligned}
$$

So, finally,

$$w'' + 2i|f'|^2 w' = \mathrm{grad}_w(U|f'|^2)$$

and the proposition is proved. $\square$

Local regularization of the RTBP: Levi-Civita transformation

In order to deal with the regularized equations, we must choose a particular $f(w)$.
Since we have two sungularities: $P_0(\mu, 0)$, $P_1(\mu - 1, 0)$ we will consider a transformation for each singularity

(a) The transformation $z = f(w) = \mu + w^2$ regularizes the singularity at P_0.

(b) The transformation $z = f(w) = \mu - 1 + w^2$ regularizes the singularity at P_1.

So both transformations are called 'local' since choosing one of them we just eliminate one of both singularities.

Example 1.

$z = f(w) = \mu + w^2$, $\qquad z = x + iy \qquad w = u + iv$
$\dfrac{dt}{ds} = |f'(w)|^2 = 4(u^2 + v^2)$
See the geometry of the change of variables z, w in [1].
f transforms

$$
\begin{array}{ccc}
z & & w \\
P_0(\mu,0) & \longrightarrow & (0,0) \\
P_1(\mu-1,0) & \longrightarrow & w_{1,2} = \pm i \\
(0,0) & \longrightarrow & w_{1,2} = \pm i\sqrt{\mu} \\
(x,y) & \longrightarrow & (u,v)
\end{array}
$$

$$u = \pm\sqrt{\frac{(x-\mu) + \sqrt{(x-\mu)^2 + y^2}}{2}}, \qquad v = \frac{y}{2n}.$$

The regularized equation?

$$w'' + 2i|f'|^2 w' = \mathrm{grad}_w(U|f'|^2)$$

$$
\begin{aligned}
U &= \frac{1}{2}\left[(1-\mu)r_0^2 + \mu r_1^2\right] + \frac{1-\mu}{r_0} + \frac{\mu}{r_1} - \frac{C}{2} = \\
&= \frac{1}{2}\left[(1-\mu)|w|^4 + \mu|1+w^2|^2\right] + \frac{1-\mu}{|w|^2} + \frac{\mu}{|1+w^2|} - \frac{C}{2}
\end{aligned}
$$

$$\begin{array}{l} \uparrow \\ r_0 = |z-\mu| = |w^2| \\ r_1 = |z-\mu+1| = |1+w^2| \end{array}$$

Since $|f'|^2 = 4(u^2+v^2)$, we have

$$u'' + iv'' + 8i(u^2+v^2)(u'+iv') = (4U(u^2+v^2))_u + i(4U(u^2+v^2))_v \Longleftrightarrow$$
$$\Longleftrightarrow \begin{cases} u'' - 8(u^2+v^2)v' = (4U(u^2+v^2))_u \\ v'' + 8(u^2+v^2)u' = (4U(u^2+v^2))_v \end{cases}$$

where

$$4U(u^2+v^2) = 2(u^2+v^2)\left[(1-\mu)\underbrace{(u^4+v^4+2u^2v^2)}_{(u^2+v^2)} + \mu\left\{(1+u^2-v^2)^2+4u^2v^2\right\}\right] +$$
$$+4(1-\mu) + \frac{4\mu(u^2+v^2)}{\sqrt{(1+u^2-v^2)^2+4u^2v^2}} - 2C(u^2+v^2)$$

and $(4U(u^2+v^2))_u$, $(4U(u^2+v^2))_v$ become singular only at $P_1 : \pm i$.
In particular, the velocities at P_0, P_1 are:

- At P_0: $\quad r_0 = 0, \quad z = \mu, \quad w = 0$

$$\begin{array}{ll} |\dot z| \longrightarrow \infty & (\text{since} \quad |\dot z| = 2U) \\ |w'| = 2\sqrt{2(1-\mu)} & (\text{since} \quad |w'|^2 = 2|f'|^2 U = 8(1-\mu)) \end{array}$$

- At P_1: $\quad r_1 = 0, \quad z = \mu - 1, \quad w = \pm i$

$$\begin{array}{ll} |\dot z| \longrightarrow \infty & \\ |w'| \longrightarrow \infty & (\text{since} \quad |w'|^2 = 2|f'|^2 U \longrightarrow \infty) \end{array}$$

Example 2. Similarly for P_1.
Comments.

1. Global regularizations: Birkhoff, Thiele-Burran, Lemaître. (see [1])
2. In the spatial RTBP, the generalization of the Levi-Civita coordinates was done by Kustaanheimo and Stiefel, the so called KS coordinates.
3. Reference [1]: "The theory of orbits" (V. Szebehely) and references therein.