The Pólya-Tchebotaröv problem and the Bloch-Landau constant

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INFORMAL SEMINAR

April 2nd, 2009

UNIVERSITAT DE BARCELONA

The Pólya-Tchebotaröv problem, 1929

Problem (1)

Given a finite number of points $E := \{a_1, \dots, a_n\} \subset \mathbb{C}$, find the continuum K with minimal capacity such that $E \subset K$.

Definition (Capacity)

Let $K \subset \mathbb{C}$ be a compact set.

$$cap(E) := \sup \left\{ |f'(\infty)| : f \in \mathcal{H}ol(\mathbb{C} \setminus K), \|f\|_\infty \leq 1, f(\infty) = 0 \right\}.$$

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Problem (2)

Given a finite number of points $E := \{a_1, \ldots, a_n\} \subset \mathbb{C} \setminus \{0\}$ *find a conformal map* $f : \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{C} \setminus E$ *such that* $f(0) = 0$ *and* $|f'(0)|$ *is maximal.*

Laurentiev's Theorem

Theorem (Laurentiev, '34)

If $E = \{a_1, \ldots, a_n\} \subset \mathbb{C}$, there exists a unique extremal domain $\Omega = f(\mathbb{D})$ for the problem 2 such that:

1. $\mathbb{C} = \Omega \cup \Gamma$ ($\Gamma := \partial\Omega$).
2. *The boundary Γ consists of finitely many simple arcs of analytic curves.*
3. *To any arc $\alpha\beta$ consisting of regular points of Γ there correspond under the conformal mapping f^{-1} two arcs of the same length on the unit circle.*

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Theorem (Goluzin, 1946)

Let $a_1, \ldots, a_n$ be arbitrary given points in $\mathbb{C}$. Let K be the extremal continuum for Problem 1. Then K is the union of the closures of all critical trajectories of the quadratic differential

$$Q(z)dz^2 = -\frac{\prod_{l=1}^{n-2}(z - b_l)}{\prod_{k=1}^{n}(z - a_k)} dz^2$$

where b_l are some unknown parameters. The extremal function $g : \mathbb{C}_\infty \setminus \mathbb{D} \to \mathbb{C}_\infty \setminus \{a_1, \ldots, a_n\}$ ($g(\infty) = \infty$) satisfy

$$(zg'(z))^2 = \frac{\prod_{i=1}^{n}(g(z) - a_i)}{\prod_{j=1}^{n-2}(g(z) - b_j)}.$$

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1. Kuzmina in 1982 computes the extremal domain in the case of three points.
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Notations

Let Ω_n be the extremal domain for the Problem 2 in case of $n+1$ points $\{a_1, \ldots, a_n, \infty\}$. Let $f : \mathbb{D} \to \Omega_n$ be the conformal map such that $f(0) = 0$. f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^2 = C\frac{\prod_{i=1}^{n}(f(z)-a_i)}{\prod_{j=1}^{n-1}(f(z)-b_j)}, \tag{1}$$

where b_j are unknown and $C = \frac{\prod_{l=1}^{n-1}(-b_l)}{\prod_{k=1}^{n}(-a_k)}$.

The code can be downloaded from http://www.maia.ub.es/cag/code/tchebotarev/.

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Case of 3 points

Assume that we have $a_1, a_2 \neq 0$ and $a_3 = \infty$. Without loss of generality we will always assume that $f(1) = \infty$.

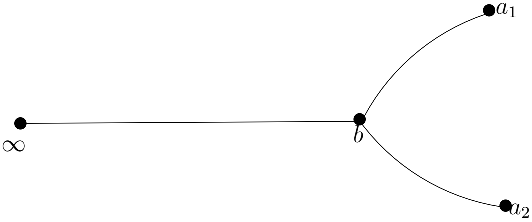

Figure: Sketch of the extremal compact for three points

We only have one unknown parameter b:

$$f'(z)^2 = C\frac{(f(z)-a_1)(f(z)-a_2)}{f(z)-b}\frac{f(z)^2}{z^2} \qquad (2)$$

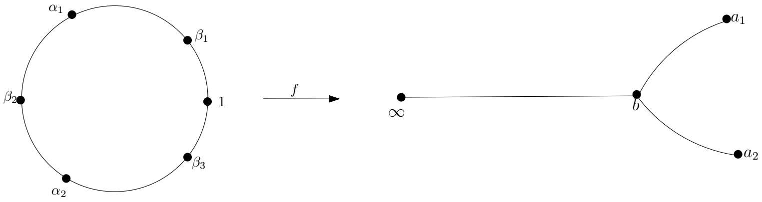

Figure: Configuration for $n = 3$

where $f(0) = 0$.
Note that $f(e^{i\alpha_i}) = a_i$ for $i = 1, 2$ and $f(e^{i\beta_i}) = b$ for $i = 1, 2, 3$.

We have 6 real unknown parameters in our problem: $\mathrm{Re}(f'(0)), \mathrm{Im}(f'(0)), \mathrm{Re}(b), \mathrm{Im}(b), \beta_1, \beta_2$ and we can impose the following three complex equations

$$\begin{cases} f(e^{i\beta_1/2}) = f(e^{-i\beta_1/2}). \\ f(e^{i(\alpha_1+\beta_1)/2}) = f(e^{i(\alpha_1+\beta_2)/2}). \\ f(e^{i(\alpha_2+\beta_2)/2}) = f(e^{i(\alpha_2+\beta_3)/2}). \end{cases}$$

To impose the equations we need to evaluate $f(e^{i\gamma})$ for any $\gamma \in [0, 2\pi) \setminus \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\}$. For that, denote $z(t) = f(te^{i\gamma})$. We know $z(0) = 0$ and $z'(0) = f'(0)e^{i\alpha}$. Note that $z(1) = f(e^{i\alpha})$.

$$z'(t)^2 = C\frac{(z(t) - a_1)(z(t) - a_2)}{(z(t) - b)}\frac{z(t)^2}{t^2} \quad (3)$$

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We used the Taylor integration method which allows us to integrate the singularity in $t = 0$.

As f is conformal, we know that $z(t) = z_1 t + z_2 t^2 + \ldots$, where $z_1 = f'(0)e^{i\gamma}$.

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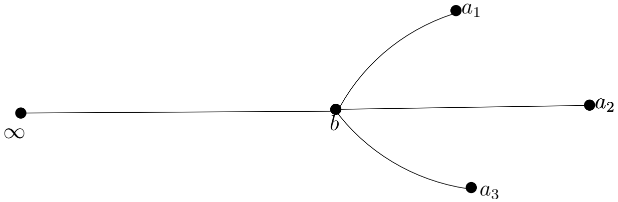
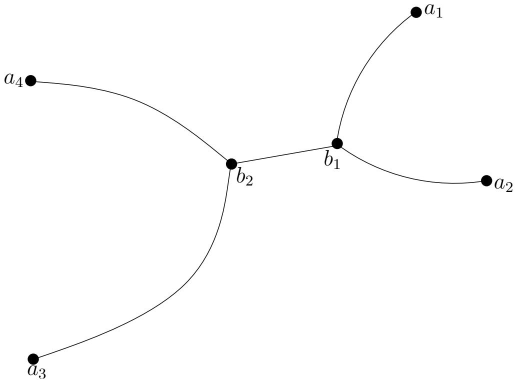
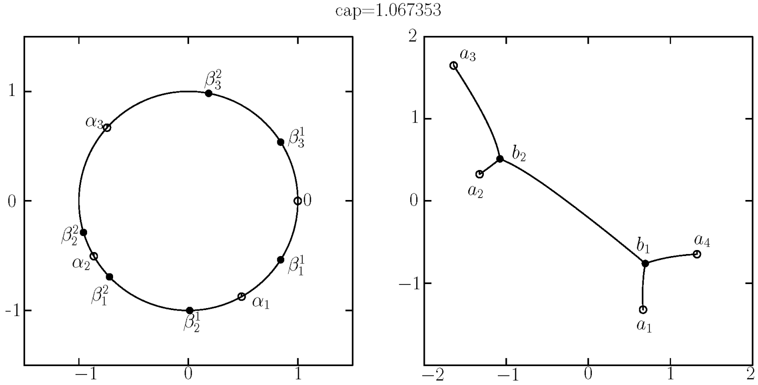
Case of 4 points

The extremal domain can be of two types. If two of the points are symmetric respect to the line through the other two points (there is an explicit solution given by Fedorov



General case





Conformal map $g : \mathbb{D}^c \to \Omega$, $g(\infty) = \infty$ where $g(e^{i\beta_k^j}) = b_j$ ($j = 1, 2$, $k = 1, 2, 3$), $g(e^{i\alpha_i}) = a_i$ ($i = 1, 2, 3$) and $g(1) = a_4$.

Assume that $a_4 = \infty$ and $a_i \neq 0$.

$$f'(z)^2 = C\frac{(f(z)-a_1)(f(z)-a_2)(f(z)-a_3)}{(f(z)-b_1)(f(z)-b_2)}\frac{f(z)^2}{z^2}. \qquad (4)$$

We can reduce the number of unknown parameters:
$\beta_3^1 = 2\pi - \beta_1^1$, $\beta_3^2 = \beta_3^1 - (\beta_1^2 - \beta_2^1)$. Unknown values $f'(0), b_1, b_2, \beta_1^1, \beta_2^1, \beta_1^2, \beta_2^2$.
We need a system of 10 real equations (5 complex equations)

$$\begin{cases} f(e^{i\beta_1^1/2}) = f(e^{-i\beta_1^1/2}) \\ f(e^{i(\alpha_1+\beta_1^1)/2}) = f(e^{i(\alpha_1+\beta_2^1)/2}) \\ f(e^{i(\beta_2^1+\beta_1^2)/2}) = f(e^{i(\beta_3^2+\beta_3^1)/2}) \\ f(e^{i(\alpha_2+\beta_1^2)/2}) = f(e^{i(\alpha_2+\beta_2^2)/2}) \\ f(e^{i(\alpha_3+\beta_2^2)/2}) = f(e^{i(\alpha_3+\beta_3^2)/2}) \end{cases}$$

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Case of 6 points (with symmetry)

We have $a_1, a_2, \ldots, a_6 = \infty$, $a_3 \in \mathbb{R}$ and $a_5 = \bar{a}_1, a_4 = \bar{a}_2$. The extremal compact may be of two types

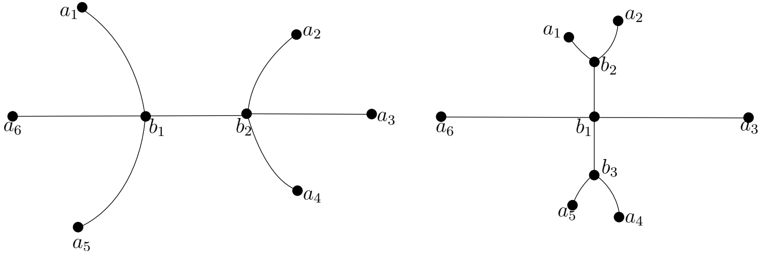

Figure: Structure of the extremal domains for $n = 6$ with symmetry

We can do some reductions to get a system of equation with less dimension.

1. First configuration: $a_3 \in \mathbb{R}$ ($\rightarrow b_1, b_2 \in \mathbb{R}$).
2. Second configuration: $a_3 \in \mathbb{R}$ ($\rightarrow b_1 \in \mathbb{R} \rightarrow b_3 = \bar{b_2}$).

By symmetry, $f'(0) \in \mathbb{R}$ and $\alpha_3 = \pi$.

First configuration

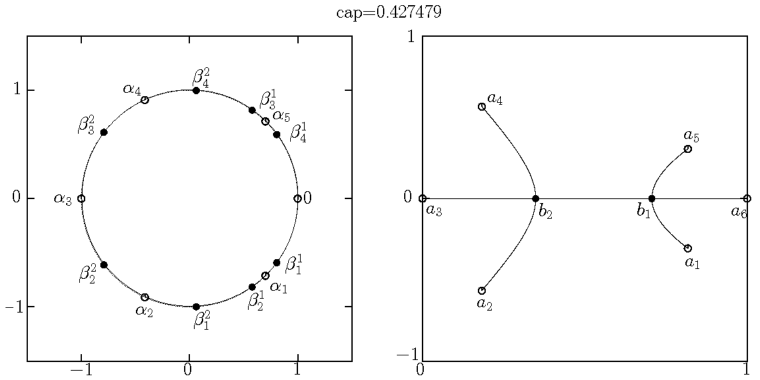

Figure: Extremal domain for $n = 6$ with symmetry (configuration 1)

Configuration: 0 β_1^1 α_1 β_2^1 β_1^2 α_2 β_2^2 α_3 β_3^2 α_4 β_4^2 β_3^1 α_5 β_4^1 2π

First configuration

We have 7 real unknown parameters: $\mathrm{Re}(f'(0))$, $\mathrm{Re}(b_1)$, $\mathrm{Re}(b_2)$, β_1^1, β_2^1, β_1^2, β_1^2.

$$\begin{cases} \mathrm{Im}(f(e^{i\beta_1^1/2.0})) = 0 \\ f(e^{i(\alpha_1+\beta_1^1)/2}) = f(e^{i(\alpha_1+\beta_2^1)/2}) \\ \mathrm{Im}(f(e^{i(\beta_2^1+\beta_1^2)/2})) = 0 \\ f(e^{i(\alpha_2+\beta_1^2)/2}) = f(e^{i(\alpha_2+\beta_2^2)/2}) \\ \mathrm{Im}(f(e^{i(\alpha_3+\beta_2^2)/2})) = 0 \end{cases}$$

Second configuration

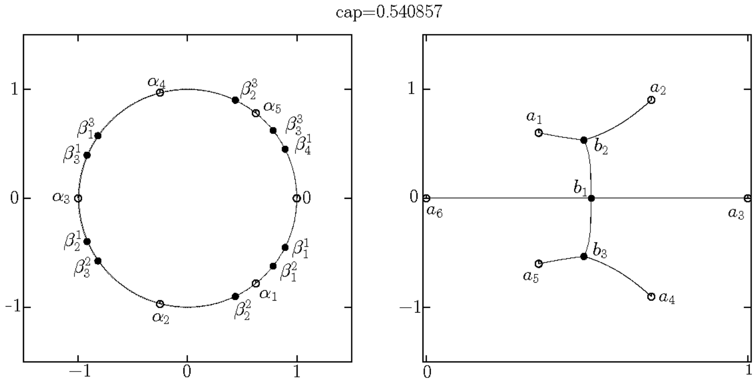

Figure: Extremal domain for $n = 6$ with symmetry (configuration 2)

$0\ \beta_1^1\ \beta_1^2\ \alpha_1\ \beta_2^2\ \alpha_2\ \beta_3^2\ \beta_2^1\ \alpha_3\ \beta_3^1\ \beta_1^3\ \alpha_4\ \beta_2^3\ \alpha_5\ \beta_3^3\ \beta_4^1\ 2\pi$

Second configuration

We have 8 real unknown parameters: $\mathrm{Re}(f'(0))$, $\mathrm{Re}(b_1)$, $\mathrm{Re}(b_2)$, $\mathrm{Im}(b_2)$, β_1^1, β_1^2, β_2^2,β_3^2.

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\begin{cases}
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\end{cases}
$$

Remark

In the implementation of the method, we found a problem when the distance between the arcs on the unit circle is very small, we can't integrate properly the differential equation because we are near the poles b_i. However this problem can solved by a change of variables.

The fundamental frequency of a drum

Theorem (Makai, 1965)

Let $D \subset \mathbb{C}$ be a simply connected domain. Let R_D be the inradius of D and let λ_D be the first eigenvalue for the Laplacian in D. There is a universal constant a such that

$$\lambda_D \geq \frac{a}{R_D^2}. \tag{5}$$

Makai's proof also shows that $1/4 \leq a < \pi^2/4 = 4.9348$.
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The expected lifetime of a Brownian motion

Let B_t be the Brownian motion in D, $\tau_D = \inf\{t > 0 : B_t \notin D\}$ be the exit time of B_t from D and $E_z(\tau_D)$ the expectation of τ_D.
It is known that there is a universal constant b such that, whenever D is a planar simply connected domain,

$$\sup_{z \in D} E_z(\tau_D) \leq b R_D^2. \quad (6)$$

It is known that $1.584 < b < 3.228$.

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The univalent Bloch-Landau constant

If f is an analytic and one to one mapping from the unit disk, then there exists a universal constant $\mathcal{U}$ such that

$$R_{f(\mathbb{D})} \geq \mathcal{U}|f'(0)|. \tag{7}$$

This means that the image of the unit disk under any conformal map f contains disks of radius less that $\mathcal{U}|f'(0)|$.

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Theorem (Koebe $1/4$)

If f belongs to $\mathcal{S}$ (f univalent in $\mathbb{D}$, normalized with $f(0) = 0$ and $f'(0) = 1$) then there is a disk $D(0, 1/4) \subset f(\mathbb{D})$. The radius $1/4$ cannot be improved. The function $f(z) = z/(1-z)^2$ is extremal.

This implies $\mathcal{U} \geq 1/4$. The best value of $\mathcal{U}$ is known as the univalent or schlicht Bloch-Landau constant.

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We can reformulate this problem in terms of the density of the hyperbolic metric.

If f is a conformal mapping from the unit disc such that $f(0) = z$ then the density of the hyperbolic metric is $\sigma(z; D) = 1/|f'(0)|$.
So we have the following inequality

$$\sigma_D := \inf_{z \in D} \sigma(z; D) \geq \frac{c}{R_D}. \tag{8}$$

where $c := \mathcal{U}$ (introduced by Landau in 1929).

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Lower bounds for $\mathcal{U}$

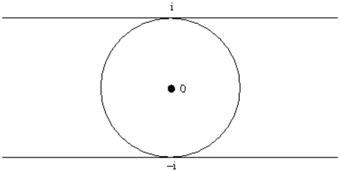
E. Landau	$\mathcal{U} > 0.566$	Mathematische Zeitschrift, 1929
E. Reich	$\mathcal{U} > 0.569$	PAMS, 1956
J. Jenkins	$\mathcal{U} > 0.5705$	J. Math Mech, 1961
S. Toppila	$\mathcal{U} > 0.5708$	Finnish Annals, 1968
J. Jenkins	$\mathcal{U} > 0.57088$	Indiana, 1998
X. Chengji	$\mathcal{U} > 0.570884$	J. Nanjing 1999

The upper bounds

To get an upper bound we construct an extremal domain Ω with inradius $R_\Omega = 1$ and we compute the conformal representation $f_\Omega : \mathbb{D} \to \Omega$, then $\mathcal{U} \leq 1/|f'_\Omega(0)|$. We assume always that $f(0) = 0$.

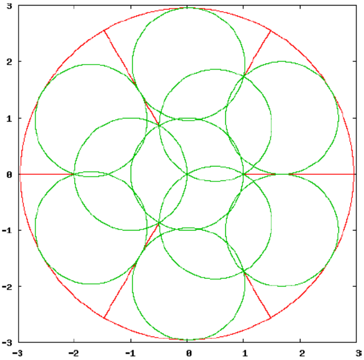
Szego, 1923

$$\mathcal{U} \leq 0.78539$$



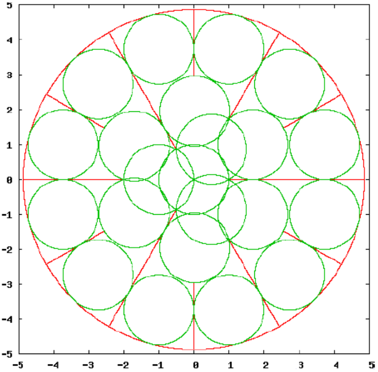
Robinson, 1935

$$\mathcal{U} \leq 0.65779$$



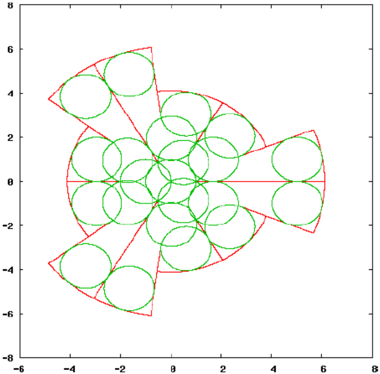
Goodman, 1945

$$\mathcal{U} \leq 0.65647$$



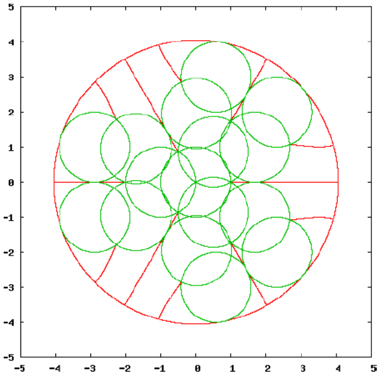
Beller and Hummel, 1985

$$\mathcal{U} \leq 0.6564155$$

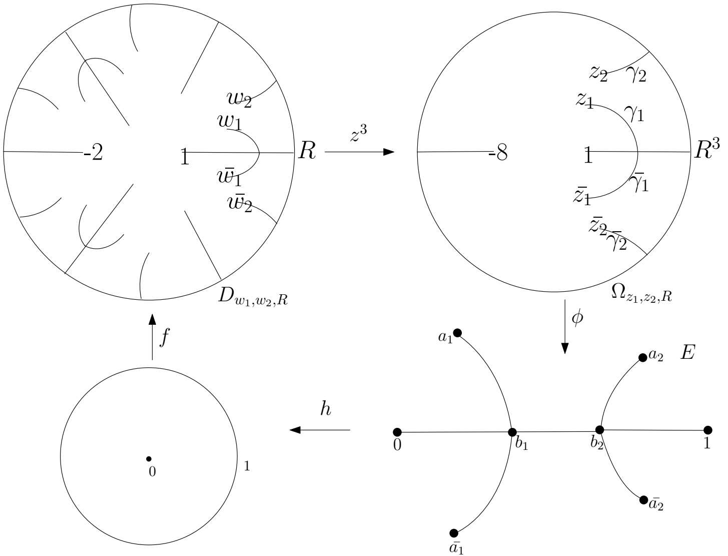


Carroll-Ortega, 2008

$$\mathcal{U} \leq 0.65639361315219$$



Relation with the Pólya-Tchebotaröv problem



Construction of the domain $D_{w_1,w_2,R}$

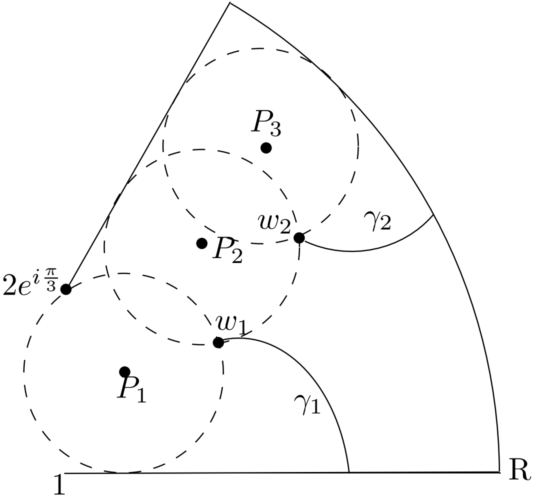


Figure: Election of w_1 and w_2

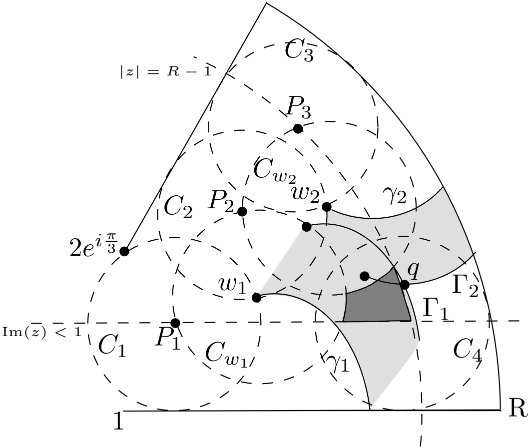


Figure: Prohibited zones

Results

We have computed the bounds for all three problems explained before.

To construct the point w_1 we move on the real axis $x >= 1 + \sqrt{2\sqrt{3} - 3}$ and define the point P_2 and then w_1.

Given x, first find the biggest R such that $|q| = 1 - R$, then compute the bounds of the constants.

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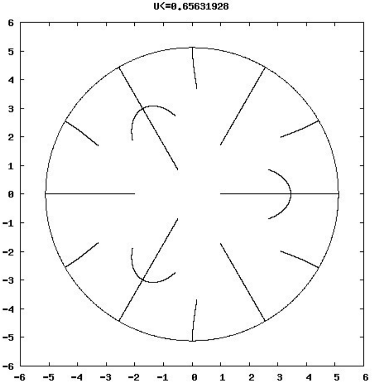
Bloch-Landau constant

$x = 2.13837799965243$ and $R = 5.1195152501$ and
$\mathcal{U} \leq 0.656319277272$ (≤ 0.65639361315219)



Constants *a* and *b*.

1. Improved upper bound for the fundamental frequency has been found for $x = 2.1282995811037759$ and $R = 5.10223601895443$ and it is

$$a \leq 2.0907934752309 (< 2.13)$$

2. Improved lower bound for the expected life time of a Brownian motion has been found for $x = 2.174447128952$ and $R = 5.1836816989$ and it is

$$b \geq 1.670724582110 (> 1.584)$$

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Thank You!