Homoclinic and Heteroclinic Motions in Quantum Dynamics

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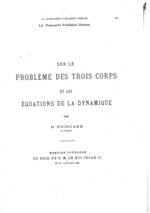
Stability and Instability in Mechanical Systems: Applications and Numerical Tools
Barcelona, 1 December 2008

Outline

Outline

UAM UNIVERSIDAD AUTONOMA DE MADRID

- In his pioneering work on chaos Poincaré showed the importance of

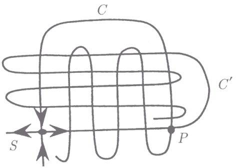


5.2. Poincaré's Published Memoir

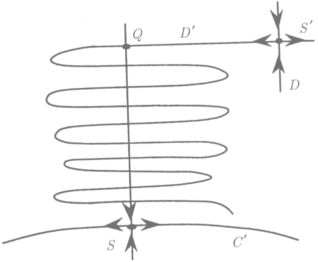
SUR LE
PROBLÈME DES TROIS CORPS
ET LES
ÉQUATIONS DE LA DYNAMIQUE
PAR
H. POINCARÉ

MÉMOIRE COURONNÉ
DU PRIX DE S. M. LE ROI OSCAR II

- Periodic orbits
- Homoclinic solutions
- Heteroclinic solutions



Homoclinic solution



Heteroclinic solution

In this talk, we will discuss the importance of:

- Periodic orbits
- Homoclinic motions
- Heteroclinic motions

in Quantum Mechanics

Outline

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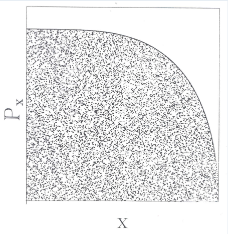
Model: Quartic oscillator

- $H = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}x^2y^2 + \frac{\varepsilon}{4}(x^4 + y^4), \qquad \varepsilon = 0.01$
- Smooth, homogeneous potential
- Mechanical similarity
  $\frac{q}{q_0} = \left(\frac{E}{E_0}\right)^{1/4}, \frac{P}{P_0} = \left(\frac{E}{E_0}\right)^{1/2}, \frac{S}{S_0} = \left(\frac{E}{E_0}\right)^{3/4}, \frac{T}{T_0} = \left(\frac{E}{E_0}\right)^{-1/4}$
  Free from hassles due to phase space evolution (bif's)

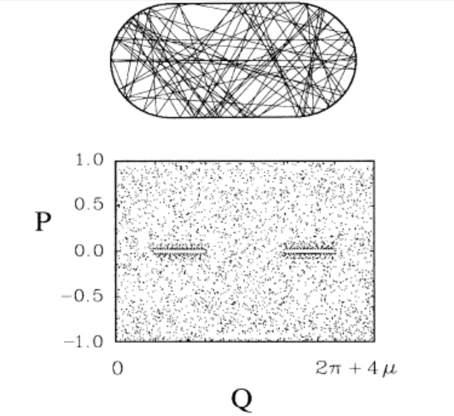
SOS: $y = 0, P_y > 0$



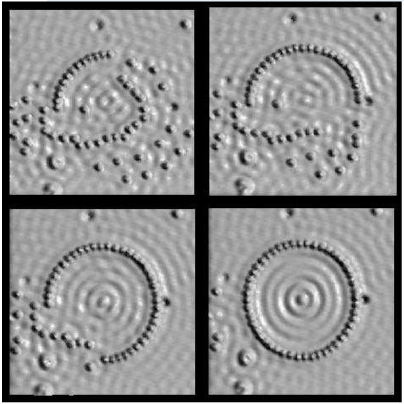
- Very chaotic dynamics
- Thought hyperbolic for $\varepsilon \to 0$
- Dahlqvist and Russberg (1990) found POs for $\varepsilon = 0$
- Also Waterland el at. for $\varepsilon = 1/240$

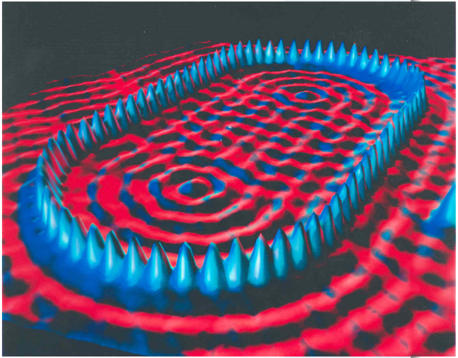
Model: Billiards

- Bunimovitch stadium billiard
- Hyperbolic dynamics



Billiards: Models in Nanotechnology

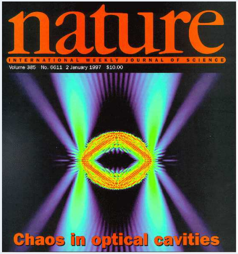


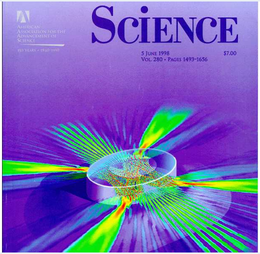


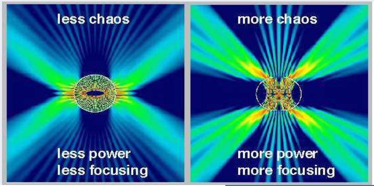
Eigler

Billiards: Models for Microcavity Lasers

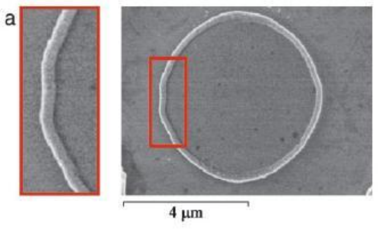
A. Douglas Stone, 1997

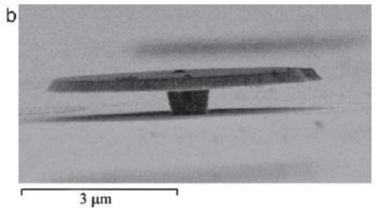






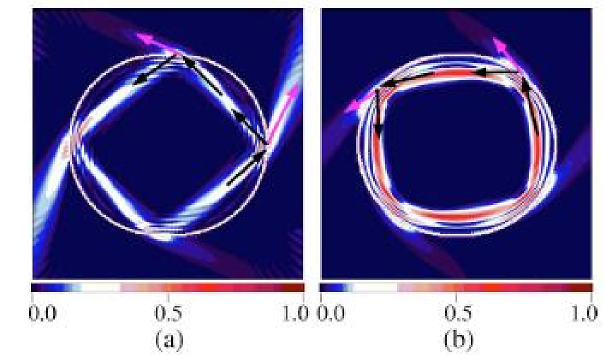
Microdisk laser, Douglas Stone, PNAS, 2004





Top and side view of a GaAs microdisk ($\sim$5.2μm diameter) on top of an $Al_{0.7}Ga_{0.3}$ pedestal.
A thin InAs quantum well layer in the middle layer serves as active medium.
Image obtained with a scanning electron microscope.

Directional Laser emission



(a) (b)

Directional laser emission has direct applications in optical communications and optoelectronics

More on microlasers ...

PRL **100**, 174102 (2008) PHYSICAL REVIEW LETTERS week ending 2 MAY 2008

Uncertainty-Limited Turnstile Transport in Deformed Microcavities

Jeong-Bo Shim,[1,2] Sang-Bum Lee,[1] Sang Wook Kim,[3] Soo-Young Lee,[1] Juhee Yang,[1] Songky Moon,[1] Jai-Hyung Lee,[1] and Kyungwon An[1,*]

[1]*School of Physics and Astronomy, Seoul National University, Seoul, 151-742, Korea*
[2]*Max-Planck Institute for Physics of Complex Systems, Nöthnitzer Strasse 38, Dresden, Germany*
[3]*Department of Physics Education, Pusan National University, Busan 609-735, Korea*
(Received 24 December 2007; published 1 May 2008)

We present both experimental and theoretical evidence for uncertainty-limited turnstile transport in deformed microcavities. As the degree of cavity deformation was increased, a secondary peak gradually emerged in the far-field emission patterns to form a double-peak structure. Our observation can be explained in terms of the interplay between turnstile transport and its suppression by the quantum

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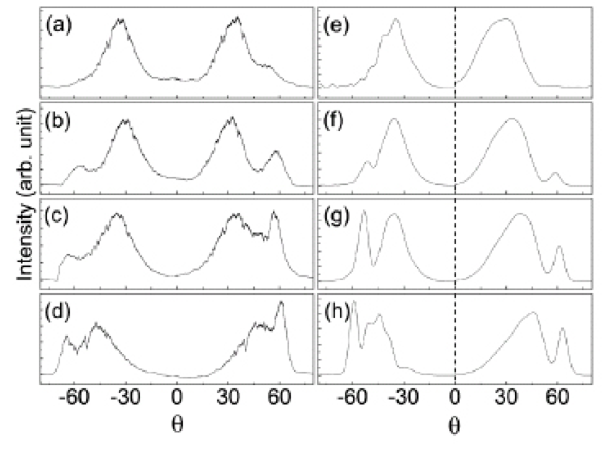
$$r(\phi) = a(1 + \eta_0 \cos 2\phi + \epsilon\eta_0 \cos 4\phi)$$

More on microlasers ...



$\eta = 0.09$

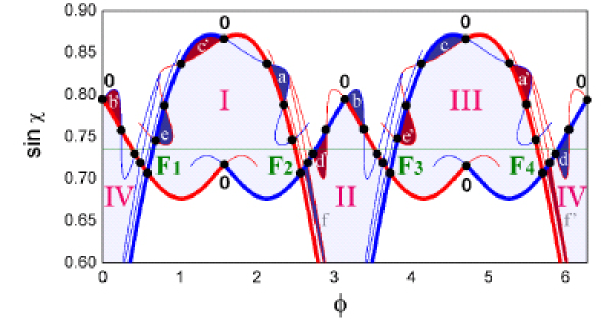
$\eta = 0.10$

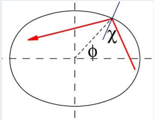
$\eta = 0.12$

$\eta = 0.16$

Exp. Th. A Th. B

More on microlasers ...



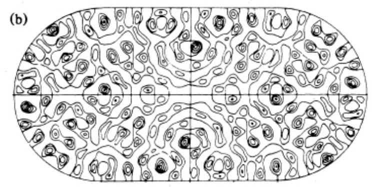


Outline

Quantum Mechanics

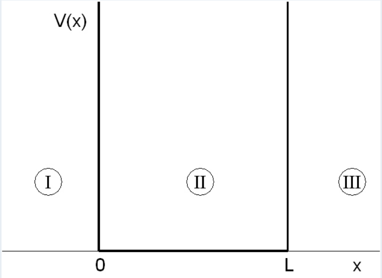
- De Broglie Hypothesis: $\lambda = \frac{h}{P} = \frac{2\pi\hbar}{P}$
- Wave function: $\psi(q,t)$, q=positions, t=time
- Stationary Schrödinger equation: with $\hat{H}\phi_n(q) = E_n\phi_n(q)$
- Heisenberg Uncertainty Principle: $\Delta q \Delta p \geq \hbar/2$ and $\Delta E \; \tau \geq \hbar/2$

Example



- Helmholtz equation: $\nabla^2 \phi_n = k_n^2 \phi_n$
  $\phi_n(boundary) = 0$

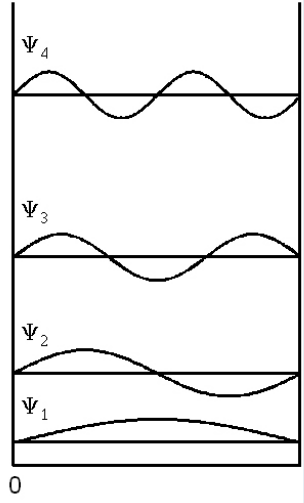
Simpler example (even trivial)



- $\psi_I = \psi_{III} = 0$
- $-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + V\psi_{II} = E\psi_{II}$
- $\frac{d^2\psi_{II}}{dx^2} + k^2\psi_{II}, \qquad k = \frac{\sqrt{2mE}}{\hbar}$
- But, don't forget the dynamics: $k = \frac{P}{\hbar}$

Solution

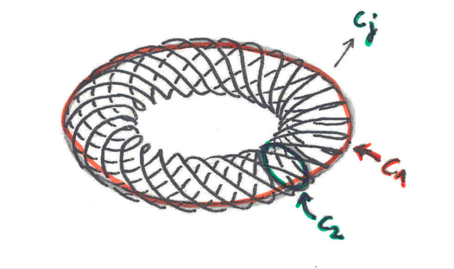
- $\psi(x) = a \sin kx + b \cos kx$
- First boundary condition:
  $\psi(0) = 0 \longrightarrow c = 0$
  $\psi = b \sin kx$
- Normalization condition:
  $\int_0^L |\psi|^2 \, dx = 1 \longrightarrow a = \sqrt{\frac{2}{L}}$
- Second boundary condition:
  $\psi(L) = 0 \longrightarrow k_n = \frac{n\pi}{L}$
- Solutions: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$



- But, don't forget the dynamics ... $k = \frac{P}{\hbar}$
- Classical action:
  $\oint P dx = 2\int_0^L P dx = 2\int_0^L k\hbar dx = 2k\hbar L = 2\frac{n\pi}{L}\hbar L = nh$
- Action is quantized in QM!

Quantization of the action. How?

- Einstein–Brillouin–Kramers (EBK) Method



$\oint_{C_j} \sum_i^N P_i \, dq_i = h \left(n_j + \frac{\alpha_j}{4}\right)$
Classical info = Quantum condition

- Associated WKB (Wentzel–Kramers–Brillouin) wave function
$\psi(q) = \sum_j A_j \, e^{iS_j(q)/\hbar}$

Phase space representations of QM

- Wigner transform (1932)
  "On the quantum corrections to statistical thermodynamics"

$$W(q,P) = \int ds\, e^{isP}\, \psi^*\left(q - \frac{s}{2}\right) \psi\left(q + \frac{s}{2}\right)$$

But ...

- $W(q, P)$ can be negative
- Why?:
  Heisenberg's uncertainty principle
- Solution: Husimi function
  - Gaussian average in cells of area $\hbar^N$
    $H(q, P) = \int \int_{\hbar^N} dq' \, dP' G_{q,P}(q', P') \, W(q', P')$
  - Coherent state representation
    $H(q, P) = \frac{1}{(2\pi\hbar)^N} |\langle \phi_{q,P} | \psi \rangle|^2$
    ϕ minimum uncertainty coherent state
    $\phi(x, y, P_x, P_y) = \left[\frac{2\alpha}{\pi}\right]^{1/4} \; e^{-\alpha(x-x_0)^2} \; e^{-\alpha(y-y_0)^2} \; e^{iP_x^0 x} \; e^{iP_y^0 y}$

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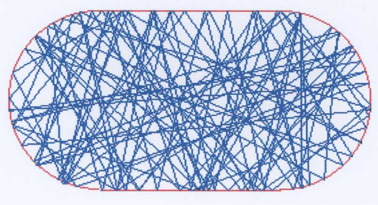
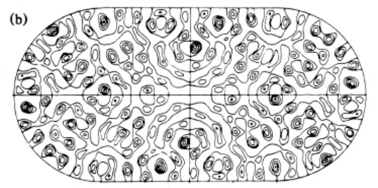
Outline

Periodic orbits in quantum mechanics: Scars

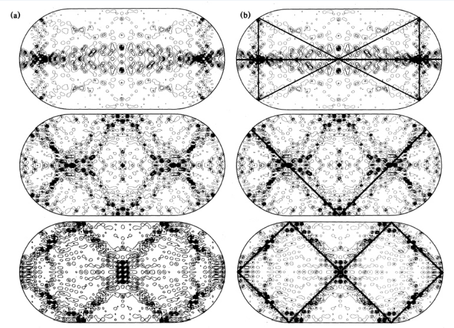
- What are scars?
  Expected: Chaotic classical dynamics $\longrightarrow$ uniformly distributed quantum density





Scarred functions

- But in numerical calculations (McDonald & Kaufman) ...



Heller in 1984 coined the term scar to name an enhanced localization of quantum probability density of certain eigenstates on classical unstable periodic orbits

Scars in Optical Fibers

VOLUME 88, NUMBER 1 PHYSICAL REVIEW LETTERS 7 JANUARY 2002

Light Scarring in an Optical Fiber

Valérie Doya, Olivier Legrand, and Fabrice Mortessagne
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Christian Miniatura
Laboratoire Ondes et Désordre, CNRS FRE 2302, 1361 route des Lucioles, Sophia-Antipolis, F-06560 Valbonne, France
(Received 31 July 2001; published 18 December 2001)

We report the first experimental study of wave scarring in an optical fiber with a noncircular cross section. This optical multimode fiber serves as a powerful tool to image waves in a system where light rays exhibit a chaotic dynamics. Far-field intensity measurements are used to provide a better identification of scars in the Fourier domain. This first experimental characterization of scarring effect in optics demonstrates the relevance of such an optical waveguide for novel experiments in wave chaos.

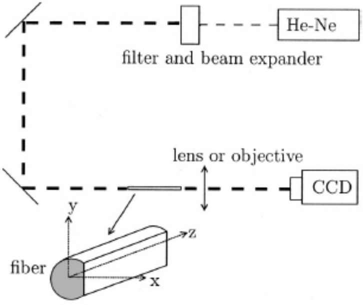
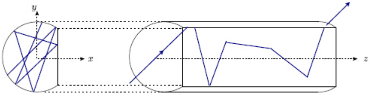


FIG. 14. Experimental setup.



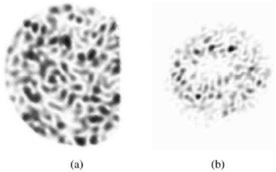
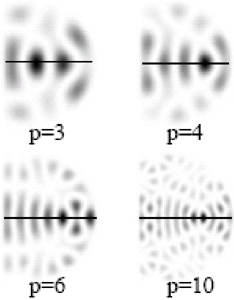


FIG. 1. Typical specklelike experimental intensity pattern at the output of a chaotic D-shaped fiber for a plane wave illumination at central wave vector $\kappa_c = 19.0R^{-1}$. (a) Near-field intensity; (b) far-field intensity.



Scars in Microcavity lasers

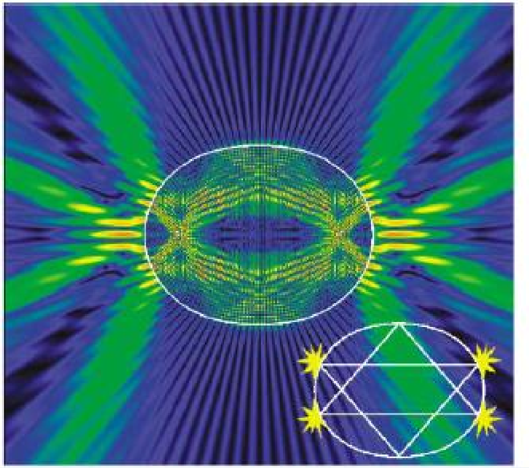
VOLUME 88, NUMBER 9 PHYSICAL REVIEW LETTERS 4 MARCH 2002

Fresnel Filtering in Lasing Emission from Scarred Modes of Wave-Chaotic Optical Resonators

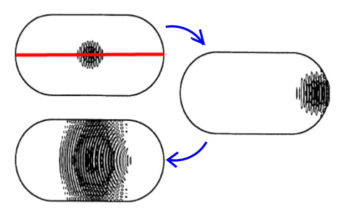
N. B. Rex, H. E. Tureci, H. G. L. Schwefel, R. K. Chang, and A. Douglas Stone
Department of Applied Physics, P.O. Box 208284, Yale University, New Haven, Connecticut 06520-8284
(Received 24 May 2001; published 19 February 2002)

We study lasing emission from asymmetric resonant cavity GaN microlasers. By comparing far-field intensity patterns with images of the microlaser we find that the lasing modes are concentrated on three-bounce unstable periodic ray orbits; i.e., the modes are scarred. The high-intensity emission directions of these scarred modes are completely different from those predicted by applying Snell's law to the ray orbit. This effect is due to the process of "Fresnel filtering" which occurs when a beam of finite angular spread is incident at the critical angle for total internal reflection.

Scars in Microcavity lasers



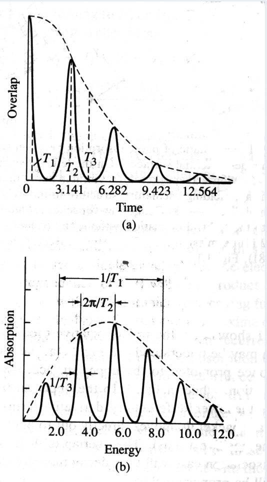
Heller's dynamical explanation for scars



Recurrences

Fourier transform between:

correlation function $C(t) = \langle \phi(0) | \phi(t) \rangle$, and corresponding spectrum $I(E) = \int \, dt \, e^{iEt/\hbar} \, C(t)$

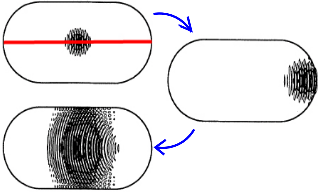


Peaks

- **Where?**
  Bohr–Sommerfeld quantization condition on the action:
  $S = \oint \; P \cdot dq = 2\pi\hbar \left(n + \frac{\alpha}{4}\right)$
- **Why?**
  Constructive interference in the WKB wavefunction
  $\psi(q) = A \; e^{iS(q)/\hbar}$

BUT ...

What happens to the density that does not come back in the recurrence along the scarring periodic orbit?



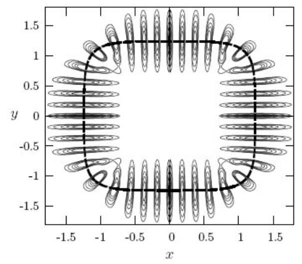
Outline

1. Introduction
   - Models
   - Tools
   - Periodic orbits in quantum mechanics: Scars
2. Constructing scar functions
3. Unveiling homoclinic motions
4. Homoclinic quantum numbers

How to systematically construct scar function

- Borondo et al., *PRL* 73, 1613 (1994) version 2007
- Wavepacket initially localized on the PO

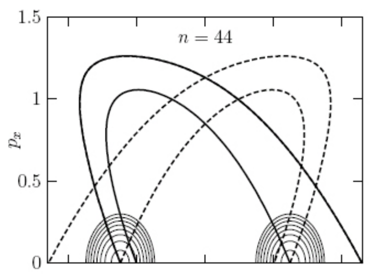
$$\psi_{\text{tube}}(x,y) = N \int_0^T dt\, e^{-\alpha_x(x-x_t)^2-\alpha_y(y-y_t)^2} \times \cos\left[S_t - \frac{\mu\pi t}{2T} + P_{xt}(x-x_t) + P_{yt}(y-y_t)\right]$$



In phase space

- Quantum SOS based on Husimi function:

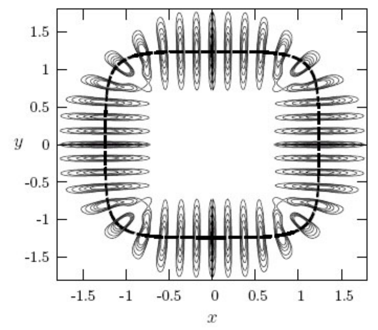
$$\mathcal{H}(x, P_x) = \left| \int_{-\infty}^{\infty} dx' \; e^{-(x-x')^2/(2\alpha_H^2) - iP_x x'} \psi(x', y' = 0) \right|^2$$



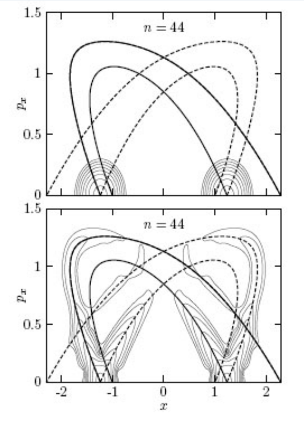
This can be improved

- Propagate $\psi_{\rm tube}(x,y)$ in time and Fourier transform at $E_{\rm BS}$

$$\psi_{\rm scar}(x,y) = N \int_{-T_E}^{T_E} dt\ \cos\left(\frac{\pi t}{2T_E}\right)\ e^{-i(\hat{H}-E_{\rm BS})t}\ \psi_{\rm tube}(x,y)$$



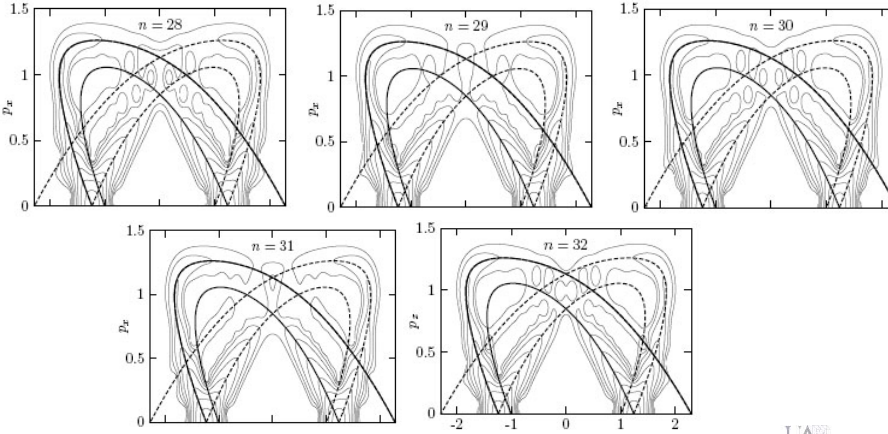
Same in phase space

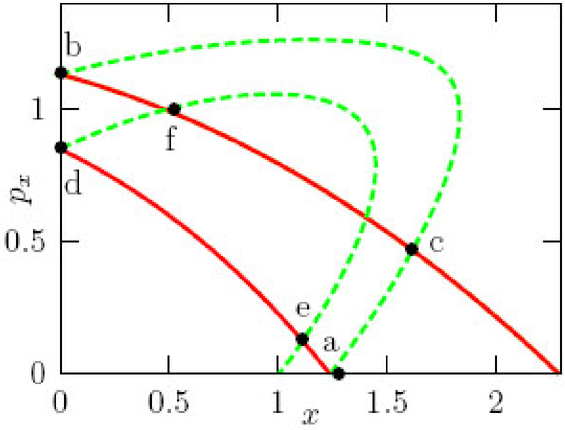


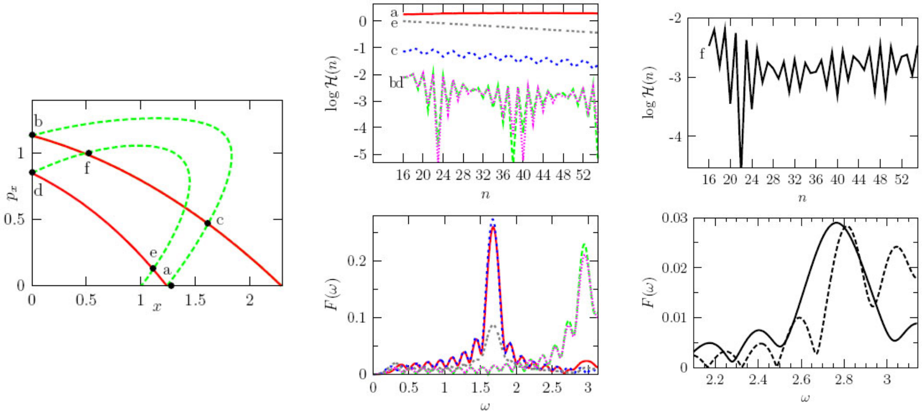
$\psi_{\mathrm{tube}}(x, y)$

$\psi_{\mathrm{scar}}(x, y)$

Outline


n = 28
n = 29
n = 30
n = 31
n = 32
p_x
1.5
1
0.5
0
-2
-1
0
1
2


b
1
f
d
p_x
0.5
c
e
a
0
0
0.5
1
x
1.5
2

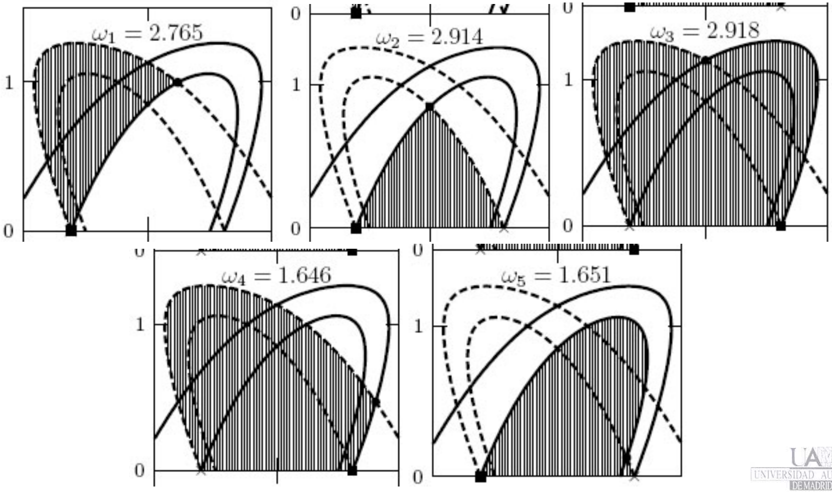


Peaks at ω = 1.67, 2.76, 2.95

Where do these frequencies come from?

Additional quantization condition for the circuits in phase space:
$S_i - \frac{\pi}{2}\nu_i = 2\pi n$

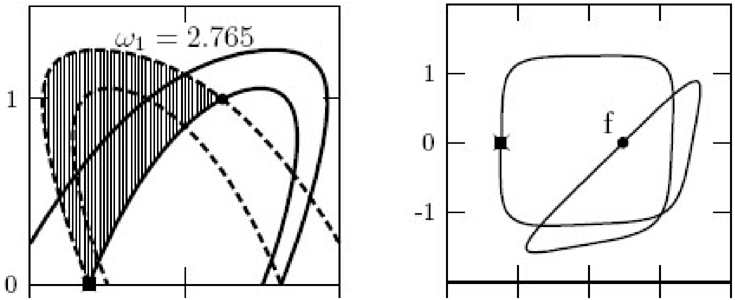
When this condition is fulfilled the recurrence along the circuit reinforce the recurrence along the scarring periodic orbit!!

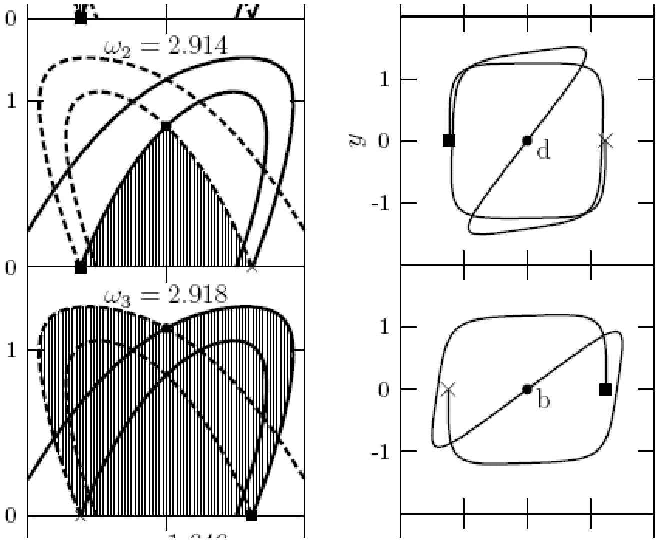

$\omega_1 = 2.765$
$\omega_2 = 2.914$
$\omega_3 = 2.918$
$\omega_4 = 1.646$
$\omega_5 = 1.651$

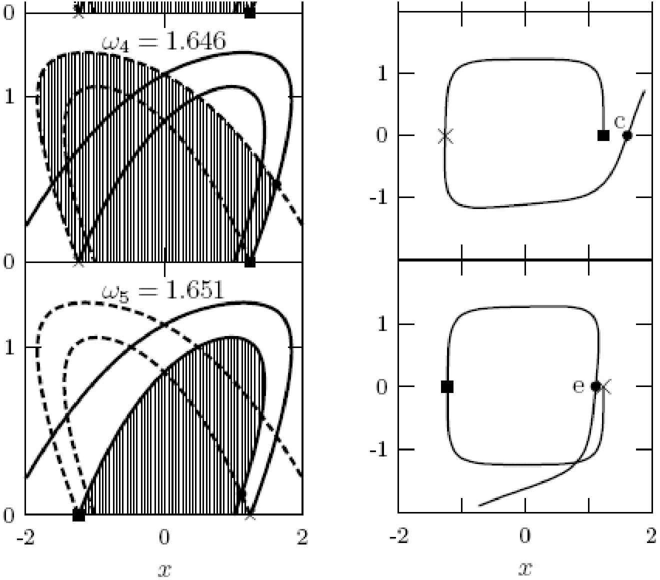
Circuit	Type	S_i	ω_i'	ω_i
1	Homoclinic	1.464	3.519	2.764
2	Heteroclinic	1.212	2.913	2.913
3	Heteroclinic	4.014	3.365	2.918
4	Heteroclinic	3.299	1.646	1.646
5	Heteroclinic	1.927	4.632	1.651

That agree reasonably well with the previously found values

ω = 1.67, 2.76, 2.95

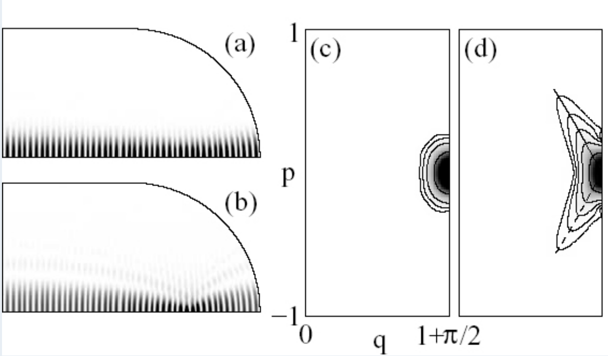

$\omega_1 = 2.765$
1
0
1
f
0
-1


$\omega_2 = 2.914$
$\omega_3 = 2.918$
0
1
0
1
0
y
1
0
-1
d
1
0
-1
b

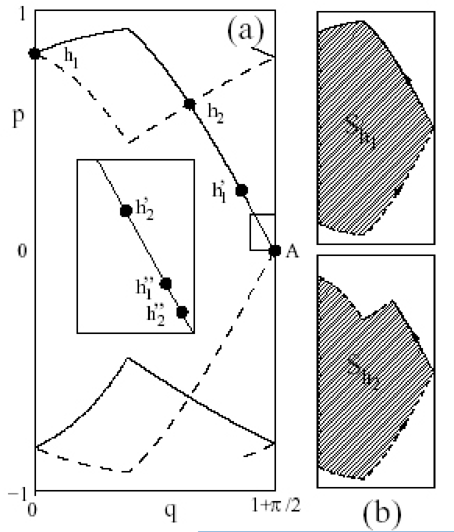

$\omega_4 = 1.646$
$\omega_5 = 1.651$
c
e
0
1
-1
-2
2
x

Homoclinic motion and wave functions
Phys. Rev. Lett. 97, 094101 (2006)

- $|\phi_{\rm scar}\rangle = \int_{-T}^{T} dt \;\cos\left(\frac{\pi t}{2T}\right)\; e^{i(E_{\rm BS}-\hat{H})t/\hbar}\,|\phi_{\rm tube}\rangle$

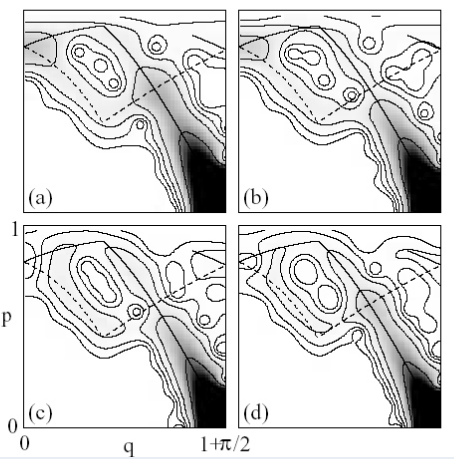


- Peaks coincide with the value of primary homoclinic areas

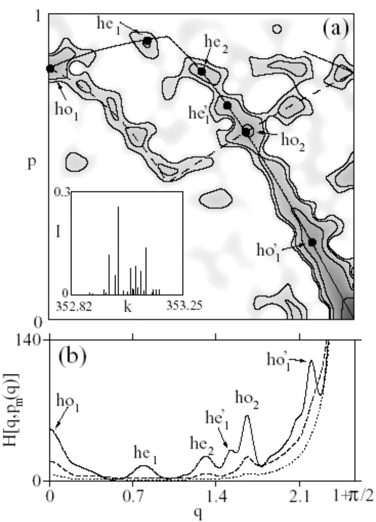


Husimis for 4 scar function with quantization/antiquantization conditions on the homoclinic torus (all quantized on the PO)

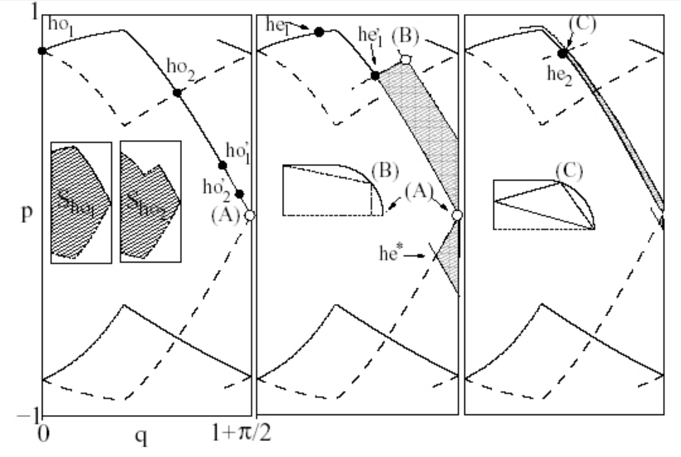
Label	n_H	k_{BS}	n_{ho_1}	n_{ho_2}
(a)	34	54.585	29.01	25.99
(b)	40	64.010	34.07	30.47
(c)	44	70.293	37.43	33.46
(d)	50	79.718	42.49	37.95



- Scar function $n = 224$
- Homoclinic quantization: $n_{h_1} = 189.01$, $n_{h_2} = 168.07$
- Extra quantization on heteroclinic orbits: $kS_{he} = 2\pi n_{he}$ $n_{he_1} = 19.00$, $n_{he_2} = 5.98$
- Husimis for $T = 0.9t_E, 1.2t_E$ and $3.3t_E$



Classical phase space



Relative relevance of the different circuits

- Why some circuits are more important than others?
- We propose a quantity to measure this:
  1. Let us consider pieces of homo or heteroclinic trajectories, as those shown before,
  2. They have initial, (x_i, P_{xi}), and ending points, (x_f, P_{x_f}), close to the fixed point, (x_F, P_{xF})
  3. Substract
  4. Apply symplectic transformation in order to write down these differences in terms of new coordinates, (u, s), living on the unstable and stable directions
  5. Define $A \equiv u_i s_j e^{\lambda T}$,
     T being the time necessary for the trajectory to go from $i \longrightarrow j$
  6. Remark: A is symplectic and canonical invariant

Circuit	Type	S_i	ω_i'	ω_i	A_i
1	Homoclinic	1.464	3.519	2.764	2.79
2	Heteroclinic	1.212	2.913	2.913	2.64
3	Heteroclinic	4.014	3.365	2.918	2.64
4	Heteroclinic	3.299	1.646	1.646	1.34
5	Heteroclinic	1.927	4.632	1.651	1.34

The agreement between A_i and the importance we found is astonishing !!!!
In the sense that

1. A takes the same value for equivalent circuits
2. More important circuits have smaller values of A

One question

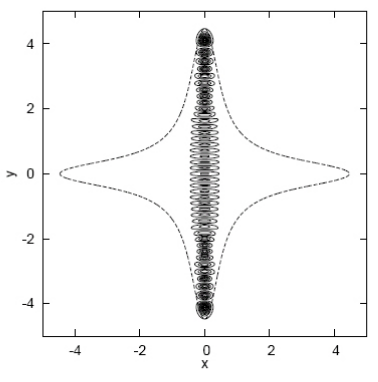
Is A_i related to the Lazutkin's invariant?

I don't know, but ... ask Ernest Fontich

Outline

Scar functions

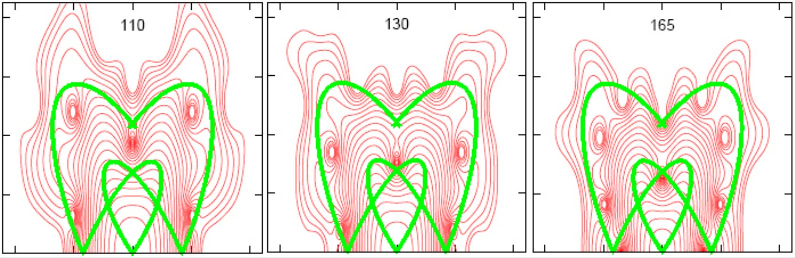
- Back to the quartic potential
- Scar functions along the vertical PO



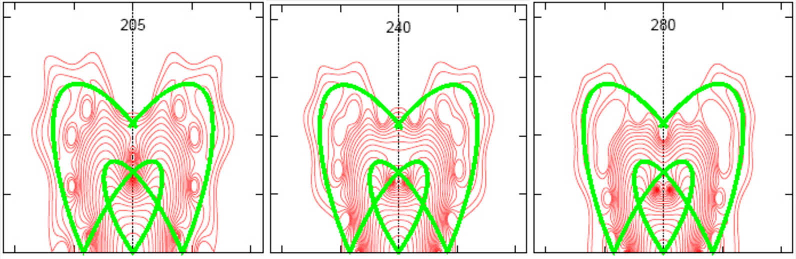
Quantum numbers

- How to compute/assign quantum numbers?
- Zeros in the Husimi function
  Leboeuf and Voros
- Zeros inside the homoclinic circuit

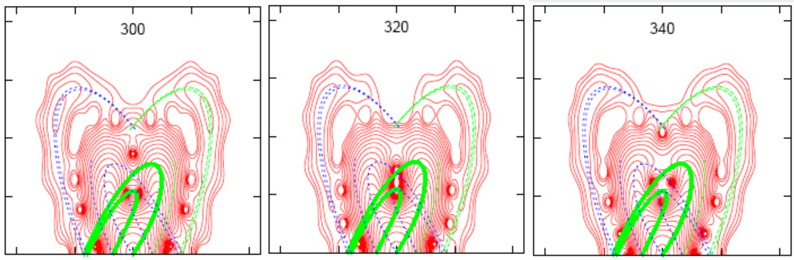
First zero in . . .



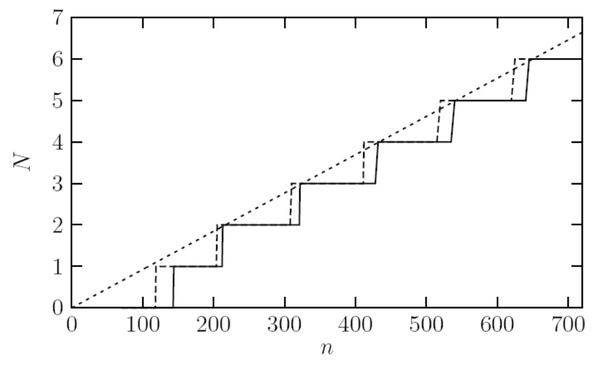
Second zero in . . .



Second zero in . . .

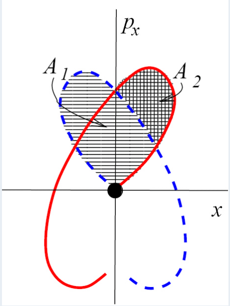


Counting the zeros in the Husimi function



Quantizing . . .

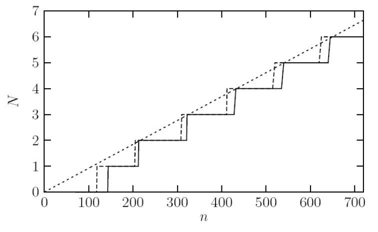
Let us make the argument quantitative . . .



$$\boxed{\frac{-A_i}{2\pi\hbar(n)} + \frac{\mu_i}{4} = m_i, \quad i = 1,2}$$

$$\begin{cases} -\dfrac{A_1 + A_2}{4\pi\hbar(n)} + \dfrac{\mu_1 + \mu_2}{8} = \overline{m} \\ \dfrac{A_2 - A_1}{2\pi\hbar(n)} - \dfrac{\mu_2 - \mu_1}{4} = \Delta m, \end{cases}$$

Quantizing . . .



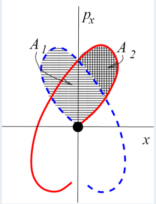


Table: Homoclinic quantum numbers $\overline{m}$

$\overline{m}$	n	Δm	n
0	43	0	141
1	163	1	713
2	282		
3	402		
4	521		
5	641		
6	760		

Thanks for your attention