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A Canonical Construction Yielding a Global View of Twistor Theory

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The construction investigated in this paper begins with an ordered, finite set of closed subgroups of some compact Lie group; from this data, the construction produces a topological space. Using a combination of fibration and cofibration techniques, it is possible to describe both the global and the local topological structure for this space. The construction yields novel, canonical decompositions of some compact manifolds (including certain spheres), as well as other interesting spaces with more exotic local topological structure. With this approach, the correspondences of *twistor theory* can be seen in their global geometric context, as a 1-parameter family of such correspondences, which canonically fit together to form S^{14}, a (constant radius) 14-dimensional sphere in a 15-dimensional Euclidean space.

1. Introduction

The aim of this paper is to introduce and then illustrate the *lattice construction* with a theoretical observation and some interesting examples. The observation shows that the three pairs of fibrations underlying the correspondences of twistor theory, may be viewed as the first part of the *lattice construction* for a particular lattice of closed subgroups of the unitary group, $U(4)$. The second part of the construction binds these fibrations into a single topological space, which is shown to be a 14-dimensional sphere. This sphere is then seen to be the global context for twistor theory in section 2.

The other goal of this paper is simply to illustrate the lattice construction, with some examples of the synthesis and analysis of certain simply-connected, compact manifolds (always without boundary). In section 3, one such 8-dimensional manifold, X^8, is obtained from the application of this canonical construction to the following lattice of Lie groups: $SU(3)$ and the two subgroups, $SU(2)$ and $SO(3)$, as well as their intersection, $SO(2)$. This manifold is referred to as $X_{(3)}$ in the more general context where $X_{(n)}$ is obtained from the lattice construction, applied to the lattice of Lie groups: $SU(n)$ and the two subgroups, $SU(n-1)$ and $SO(n)$, as well as their intersection, $SO(n-1)$. However, only $X_{(3)}$ is a closed manifold.

A point in a topological space will be called a *non-singular* point, if it has a neighborhood homeomorphic to a Euclidean space. The set of *singular* points in the space (by definition, points where the space is not locally Euclidean) will be termed the *singular subspace*. Specifically, this singular subspace for $X_{(3)}$ is empty, while for $X_{(n)}$ with $n \geq 4$, the singular subspace is the embedded sphere, $S^{2n-1} \subset X_{(n)}$. The neighborhoods of these singular points are explicitly described.

In section 4, for each $n \geq 2$, we consider the lattice: $SU(n+1)$ and two distinct subgroups, both isomorphic to $SU(n)$, and so that their intersection is $SU(n-1)$. In this case, the *lattice construction* produces a $(4n+1)$ dimensional manifold, denoted Y^{4n+1}.

Both $X_{(3)}$ and Y^{4n+1} are simply-connected, compact manifolds with the homology of a product of two spheres. Specifically, there are isomorphisms, $H_*(X_{(3)}) \cong H_*(S^3 \times S^5)$ and $H_*(Y^{4n+1}) \cong H_*(S^{2n} \times S^{2n+1})$.

An elegant observation of A. Borel [1] strongly suggested that Y^{4n+1} is homeomorphic to the real Stiefel manifold, $V_{2n+2,2}$ which consists of all orthonormal 2-frames in $\mathbb{R}^{2n+2}$. I know of no such identification for X^8.

This canonical *lattice construction* is similar to that of a finite CW-complex, constructed by skeleta of increasing dimension. The major difference is that the concept of the cell (in the skeletal filtration of a CW-complex), is replaced by the product space of a standard simplex and the appropriate homogeneous space. In section 5 we discuss some generalizations of the lattice construction, and consider briefly the spectral sequence associated with the *skeletally induced filtration* of the constructed space. In the language of category theory, A.Svensson [12] showed that the lattice construction is *functorial* and discussed some of the advantages of knowing this fact.

The simplest way to describe the construction may not be the most illuminating; however, it is the most direct, and for that reason it will be presented first. $\mathbb{R}$ and $\mathbb{C}$ will always denote the real numbers and the complex numbers, respectively. Let G be a compact Lie group, and suppose that $G_0, G_1, G_2, \ldots, G_m$ are $m+1$ closed subgroups of G. Let $\Delta_m \subset \mathbb{R}^{m+1}$ denote the standard m-simplex with *barycentric coordinates*

$b_0, b_1, b_2, \ldots, b_m$ (i.e., each $b_j \geq 0$ and $\Sigma_{j=0}^m b_j = 1$). The construction we wish to describe yields a *quotient space* of the product space

$$G \times \Delta_m = \{ (g, \beta) \mid g \in G,\ \beta = (b_0, b_1, \ldots, b_m) \}.$$

In order to produce the desired quotient space, we require the following equivalence relation, "$\sim$" on $G \times \Delta_m$.

Definition: Let $(g, \beta), (g', \beta') \in G \times \Delta_m$ be two arbitrary elements of the product space. We define the relation, $(g, \beta) \sim (g', \beta')$ to mean that (1) $\beta = \beta' = (b_0, b_1, \ldots, b_m) \in \Delta_m$ and (2) for each k ($k = 0, 1, \ldots, m$), if $b_k \neq 0$, then $g^{-1}g' \in G_k$. ∎

The quotient space, $X = (G \times \Delta_m)/(\sim)$ is then the result of applying the lattice construction to the given lattice of subgroups of G. It is this topological space, X and certain subspaces of X (together with their relation to the lattice of Lie groups) that we wish to study in this paper.

The projection onto the second factor, $G \times \Delta_m \to \Delta_m$ induces a map on the quotient space,

$$\beta : X \to \Delta_m.$$

More generally, if K is *any subcomplex* of Δ_m then $X(K) = \beta^{-1}(K) \subset X$ is filtered by skeleta. Let K^p denote the *p-skeleton* of K, and use the map β to define the *skeletally induced filtration* of the space $X(K)$ while dropping reference to K in order to simplify the notation:

$$X_p = \beta^{-1}(K^p)$$

This filtration,

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_m = X(K)$$

gives rise to a spectral sequence (in section 5), which converges to $H^*(X(K))$, the cohomology of the space $X(K)$.

We conclude this section with the observation that the most unusual feature of this spectral sequence is the way in which it begins:

$$E_1{}^{p,q} = H^{p+q}(X_p, X_{p-1}) \cong \sum_\sigma H^q(G/G[\sigma])$$

where the index for the direct sum, σ ranges over the set of p-simplices of K, and

$$G[\sigma] = \bigcap_{i=0}^{p} G_{r_i}$$

where $\{b_{r_0}, b_{r_1}, \ldots, b_{r_p}\}$ are the non-zero barycentric coordinates in the interior of the p-simplex σ. In this context, the first differential, $d_1 : E_1 \to E_1$ can be described explicitly, in terms of the above direct sum decomposition.

2. An Envelope for Twistor Theory

For each $k = 1, 2, 3$, let $U(k) \times U(4-k)$ be included in $U(4)$, in the usual way: the element of $U(k)$ is in the upper left corner of a 4×4 matrix, the element of $U(4-k)$ is in the lower right corner of the 4×4 matrix, and the remaining $2k(4-k)$ entries are zero. Now we *could* simply apply the lattice construction to the lattice of subgroups of $G = U(4)$ which consists of all seven of the subgroups that are intersections of the subgroups:

$$G_0 = U(1) \times U(3)$$
$$G_1 = U(2) \times U(2)$$
$$G_2 = U(3) \times U(1).$$

In addition to these three, the four others are:

$$G_0 \cap G_1 = U(1) \times U(1) \times U(2)$$
$$G_0 \cap G_2 = U(1) \times U(2) \times U(1)$$
$$G_1 \cap G_2 = U(2) \times U(1) \times U(1)$$
$$G_0 \cap G_1 \cap G_2 = U(1) \times U(1) \times U(1) \times U(1).$$

The seven corresponding homogeneous spaces, obtained as quotient spaces of $U(4)$, also correspond to the seven subsimplices of Δ_2 in the lattice construction of the space X.

However, there is an easier way to identify the desired space X, in this case, using the *adjoint representation* of $U(4)$ on its Lie algebra, $i\,\mathrm{Herm}(4, \mathbb{C})$, the real vector space of all 4×4, complex, anti-Hermitian matrices. The action is by conjugation, and it is represented here as the (right) conjugate action on $\mathrm{Herm}(4, \mathbb{C})$, the 16-dimensional, real vector space of all 4×4, complex, Hermitian matrices.

$$(H, U) \mapsto H^U = U^* H U : \mathrm{Herm}(4, \mathbb{C}) \times U(4) \to \mathrm{Herm}(4, \mathbb{C})$$

Let $\mathrm{Herm}_0(4, \mathbb{C}) = \{\, H \in \mathrm{Herm}(4, \mathbb{C}) \mid \mathrm{Tr}(H) = 0 \,\}$, the 15-dimensional subspace of all *traceless* matrices. Since $\mathrm{Tr}(U^* H U) = \mathrm{Tr}(H)$, $\mathrm{Herm}_0(4, \mathbb{C})$ is an *invariant subspace*.

For real number, $r > 0$, let $S(r) = \{\, H \in \mathrm{Herm}_0(4, \mathbb{C}) \mid \mathrm{Tr}(H^2) = r^2 \,\}$, which is topologically a 14-dimensional sphere, since $\mathrm{Tr}(H^2)$ is just the Hilbert-Schmidt Euclidean norm on $\mathbb{C}^{16} \cong \mathbb{R}^{32}$, restricted to the vector subspace $\mathrm{Herm}_0(4, \mathbb{C}) \cong \mathbb{R}^{15}$. If $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is the set of eigenvalues of $H \in \mathrm{Herm}_0(4, \mathbb{C})$, then $\Sigma_{i=1}^4 \lambda_i = \mathrm{Tr}(H) = 0$. If moreover $H \in S(r)$, then $\Sigma_{i=1}^4 (\lambda_i)^2 = \mathrm{Tr}(H^2) = r^2$. Since the eigenvalues are invariant under the conjugate action of $U(4)$, it follows that each sphere, $S(r)$ is an *invariant subspace*.

For the remainder of this section, let S^{14} denote $S(1)$, and consider the conjugate action of $U(4)$ on S^{14}.

$$S^{14} \times U(4) \to S^{14}$$

Each orbit contains a unique diagonal matrix with eigenvalues in non-decreasing order down the diagonal. Since the eigenvalues of any $H \in S^{14}$ can be reordered so that $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$ and $\lambda_1 < \lambda_4$, the map

$$\beta : S^{14} \to \Delta_2 : H \mapsto \beta(H) = (b_0(H), b_1(H), b_2(H))$$

is *well-defined*, where the barycentric coordinates b_k $(k = 0, 1, 2)$ are defined by

$$b_k = \frac{\lambda_{k+2} - \lambda_{k+1}}{\lambda_4 - \lambda_1}$$

Since we consider only orbits in S^{14}, the eigenvalues must satisfy

$$\sum_{i=1}^{4} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{4} (\lambda_i)^2 = 1.$$

Thus, the barycentric coordinates $\beta(H)$ determine the eigenvalues of H. In this way, the orbits of $U(4)$ in S^{14} correspond *bijectively* with the points of the standard 2-simplex, Δ_2, and we will identify this as the orbit space, with orbit-projection map β. Using three (barycentric) coordinates to describe the two dimensional simplex, has the advantage of faithfully representing all possible degeneracies of the spectrum in the simplicial structure of Δ_2.

Let $X_j = \beta^{-1}((\Delta_2)^j)$, where $j = 0, 1, 2$, and $(\Delta_2)^j$ is the $j-skeleton$ of the 2-simplex (orbit space). Thus, $X_2 = S^{14}$ is constructed from X_1, which is constructed from

$$X_0 = \Big(G/G_0\Big) \sqcup \Big(G/G_1\Big) \sqcup \Big(G/G_2\Big),$$

the *disjoint union* of the three indicated *Grassmann manifolds* (at the vertices of Δ_2).

The 1-simplex with vertices v_0 and v_1, consists of the 1-parameter family of orbits in S^{14} corresponding to the condition

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 = \lambda_4$$

(i.e., $b_2 = 0$). With the exception of the two orbits represented by the end points v_0 and v_1, the other orbits (those represented by interior points) are each homeomorphic to the compact homogeneous space $U(4) \,/\, U(1) \times U(1) \times U(2) = F(1, 1, 2)$, a 10-dimensional flag manifold called the *twistor correspondence space*. This flag manifold is the total space of two quite different fibre bundles:

$$\phi : F(1, 1, 2) \to G(1, 3) = U(4) \,/\, U(1) \times U(3) = \mathbb{C}P(3)$$

has fibre $\mathbb{C}P(2)$ while

$$\psi : F(1, 1, 2) \to G(2, 2) = U(4) \,/\, U(2) \times U(2)$$

has fibre, $\mathbb{C}P(1) = S^2$.

The various complex flag manifolds (as well as the corresponding real flag manifolds) and the fibrations between them, are explored in detail and with greater generality in [5, Chapter 2 and Appendix D] (see [6] for an announcement of results). In this section, only the one *special case* relevant to twistor theory will be described. Some of these more general results are reported in section 5.

In *twistor theory* (see [9],[10] and [13]), the two fibrations,

$$\mathbb{C}P(3) \xleftarrow{\phi} F(1, 1, 2) \xrightarrow{\psi} G(2, 2)$$

form the *twistor correspondence* between the *compact, complexified Minkowski space*, $G(2,2)$, and the twistor space, $G(1,3) = \mathbb{C}P(3)$. Thus, points in $G(2,2)$ correspond to embedded copies of $\mathbb{C}P(1) = S^2$ in $\mathbb{C}P(3)$, and points in $\mathbb{C}P(3)$ correspond to embedded copies of $\mathbb{C}P(2)$ in $G(2,2)$.

More generally, all the *correspondences* of twistor theory are given as a pair of fibre bundle projections, p and q, which have a common total space

$$A \xleftarrow{p} B \xrightarrow{q} C$$

Such a correspondence can also be represented topologically as the *double mapping cylinder*

$$A \cup_p \left(B \times [0,1]\right) \cup_q C.$$

Notice that in case $p : B \to A$ is an homeomorphism, then the double mapping cylinder is just the usual mapping cylinder of the map $q : B \to C$.

Now we attach the 1-flag cell, $F(1,1,2) \times [0,1]$ at the two ends of this cylinder, to X_0, at the two (more degenerate) orbits at v_0 and v_1. This yields

$$\mathbb{C}P(3) \cup_\phi \left(F(1,1,2) \times [0,1]\right) \cup_\psi G(2,2)$$

Similarly, the 1-simplex with vertices v_1 and v_2, represents the 1-parameter family of orbits in S^{14} corresponding to the condition

$$\lambda_1 = \lambda_2 \le \lambda_3 \le \lambda_4$$

(i.e., $b_0 = 0$). With the exception of the two orbits represented by the end points v_1 and v_2, the other represented orbits are each homeomorphic to $U(4) \,/\, U(2) \times U(1) \times U(1) = F(2,1,1)$, a 10-dimensional flag manifold called the *dual twistor correspondence space*, which is again the total space of two different fibre bundles:

$$\phi' : F(2,1,1) \to G(3,1) = U(4)/U(3) \times U(1) = \mathbb{C}P(3)$$

has fibre $\mathbb{C}P(2)$, while

$$\psi' : F(2,1,1) \to G(2,2) = U(4)/U(2) \times U(2)$$

has fibre, $\mathbb{C}P(1) = S^2$.

The two fibrations,

$$G(2,2) \xleftarrow{\psi'} F(2,1,1) \xrightarrow{\phi'} \mathbb{C}P(3)$$

comprise the *dual twistor correspondence* between the previously mentioned *compact, complexified Minkowski space* $G(2,2)$ and the *dual twistor space* $G(3,1) = \mathbb{C}P(3)$. Once again, the points in $G(2,2)$ correspond to embedded copies of $\mathbb{C}P(1) = S^2$ in $\mathbb{C}P(3)$.

Now we attach the second 1-flag cell, $F(2,1,1) \times [0,1]$ at the two ends of this cylinder, to the two (more degenerate) orbits at v_1 and v_2. The result of both operations is the space

$$\mathbb{C}P(3) \cup_\phi \left(F(1,1,2) \times [0,1]\right) \cup_\psi G(2,2) \cup_{\psi'} \left(F(2,1,1) \times [0,1]\right) \cup_{\phi'} \mathbb{C}P(3)$$

To complete the construction of X_1, the third 1-flag cell, $F(1,2,1) \times [0,1]$ must be attached to the remaining pair of orbits represented by the vertices, v_0 and v_2, both of which are $\mathbb{C}P(3)$. This 1-parameter family of orbits corresponds to the condition $b_1 = 0$, which can also be written in terms of eigenvalues as

$$\lambda_1 \leq \lambda_2 = \lambda_3 \leq \lambda_4$$

Again, the flag manifold, $U(4) \ / \ U(1) \times U(2) \times U(1) = F(1,2,1)$ is the total space of two different fibre bundles.

$$\phi'' : F(1,2,1) \to G(3,1) = U(4)/U(3) \times U(1) = \mathbb{C}P(3)$$
$$\psi'' : F(1,2,1) \to G(1,3) = U(4)/U(1) \times U(3) = \mathbb{C}P(3)$$

Both of these fibre bundle projections (which are the two attaching maps) have their fibres homeomorphic to $\mathbb{C}P(2)$, the (4-dimensional) complex projective plane. Thus, X_1 has been constructed from X_0.

Finally, the 2-flag cell, $F(1,1,1,1) \times \Delta_2$ will be attached to X_1 along its boundary, $\partial\left(F(1,1,1,1) \times \Delta_2\right) = F(1,1,1,1) \times \partial\Delta_2$, by the map,

$$F(1,1,1,1) \times \partial\Delta_2 \to X_1$$

which collapses orbits to more degenerate orbits, as appropriate, while the barycentric coordinate, $\beta \in \partial\Delta_2$ remains unchanged. Thus, $X_2 = S^{14}$ has been constructed from X_1, as promised, and we have only to locate, within S^{14}, the pair of fibre bundle projections which constitute the *ambitwistor correspondence*. In fact, the 2-flag cell, $F(1,1,1,1) \times \Delta_2$ may be viewed as a 1-parameter family of such pairs of fibre bundle projections.

$U(4) \ / \ U(1) \times U(1) \times U(1) \times U(1) = F(1,1,1,1)$ is a 12-dimensional flag manifold called the *ambitwistor correspondence space*, which is the total space of two rather different fibre bundles.

$$\lambda : F(1,1,1,1) \to G(2,2) = U(4) \ / \ U(2) \times U(2)$$
$$\mu : F(1,1,1,1) \to F(1,2,1) = U(4) \ / \ U(1) \times U(2) \times U(1)$$

In this context, $F(1,2,1)$ is called the *ambitwistor space*. Since points in the orbit space, Δ_2, bijectively represent individual orbits of $U(4)$ in S^{14}, it follows that double mapping cylinders can be used to construct (a space homeomorphic to) the subspace

$$\beta^{-1}(\nu([0,1])) \subset S^{14}$$

for any simple, continuous path, $\nu : [0,1] \to \Delta_2$ satisfying the condition that the *dimension* of the orbit $\nu(s)$ *remains unchanged* for all $0 < s < 1$ (but will decrease at both end points $\nu(0)$ and $\nu(1)$, in almost all interesting cases, though this is not part of the condition).

Many examples of such subspaces of S^{14} can be represented by straight line segments in Δ_2, satisfying the above condition on the constancy of the dimension of orbits represented by interior points of the straight line segment. The most relevant such straight line segments in Δ_2, are those with one end point at vertex v_1, and the other end point on the straight line segment (1-simplex) joining the vertices, v_0 and v_2.

We will refer to the double mapping cylinders, which are associated with twistor correspondences, by the same terms: twistor, dual twistor and ambitwistor. If β^{-1} is applied to the straight line segment (1-simplex) joining the vertices, v_0 and v_1 in Δ_2, then the resulting subspace of S^{14} will be the *twistor double mapping cylinder*. Similarly, if β^{-1} is applied to the straight line segment (1-simplex) joining the vertices, v_1 and v_2 in Δ_2, then the resulting subspace of S^{14} will be the *dual twistor double mapping cylinder*. Finally, if β^{-1} is applied to any single straight line segment joining the vertex, v_1 to an interior point of the opposite 1-simplex in Δ_2, then the resulting subspace of S^{14} will be homeomorphic to the *ambitwistor double mapping cylinder* .

In the context of algebraic geometry, each orbit of $U(4)$ in S^{14} is an algebraic variety. Replacing unitary groups by orthogonal groups (and Hermitian matrices by real symmetric matrices) throughout this construction will produce a similar decomposition (over Δ_2) of the 8-dimensional sphere, S^8. Moreover, each orbit of $O(4)$ in S^8 can be regarded as the space of *real points* in the corresponding $U(4)$-orbit in S^{14}. For details, the reader is referred to [5, Chapter 2].

3. The 8-dimensional manifold, X^8

The Lie group $G = SU(3)$ contains the two closed subgroups, $G_0 = SU(2)$ and $G_1 = SO(3)$, where the inclusion of $SU(2)$ in $SU(3)$ uses the upper left (2×2) submatrix of the (3×3) matrix, as usual, and $SO(3)$ is the subgroup consisting of those matrices of $SU(3)$ containing only real entries. Of course, the intersection $SU(2) \cap SO(3) = SO(2)$ is a circle. In this section we apply the canonical construction (described in section 1) to this lattice of subgroups. Since this lattice consists of only two subgroups and their intersection, the lattice construction yields

$$X = (G/G_0) \cup_p \Big((G/(G_0 \cap G_1)) \times [0,1] \Big) \cup_q (G/G_1).$$

In order to understand the two attaching maps, p and q (at the two ends of the cylinder), it is necessary to consider the 7-dimensional manifold, $W^7 = SU(3)/SO(2)$ and view the two attaching maps as the canonical fibre bundle projections.

$$p : W \to S^5 = SU(3)/SU(2)$$
$$q : W \to M^5 = SU(3)/SO(3)$$

Both of these fibre bundles have fibres homeomorphic to S^2, the two dimensional sphere. Thus, from just the $S^2 - bundle$, $p : W \to S^5$, it follows easily that W is a simply-connected, 7-dimensional manifold with homology isomorphic to the homology of the product space, $S^2 \times S^5$.

The homotopy exact sequence for the fibre bundle,

$$SU(3) \longrightarrow M^5 = SU(3)/SO(3)$$

with fibre $SO(3) = \mathbb{R}P(3)$, implies that M is a simply-connected, 5-dimensional manifold and $\pi_2(M) = Z_2$. The Hurewicz theorem and Poincare duality can be applied, and we find that $H_0(M) = Z$, $H_1(M) = 0$, $H_2(M) = Z_2$, $H_3(M) = 0$, $H_4(M) = 0$ and $H_5(M) = Z$. Further calculations yield $\pi_3(M) = Z_4$ and $\pi_4(M) = 0$.

Now we can define the 8-dimensional *manifold* X as the double mapping cylinder

$$X^8 = S^5 \cup_p \Big(W^7 \times [0,1] \Big) \cup_q M^5,$$

where the cylinder on W is attached at each end to the base space of the indicated fibre bundle projection. From the Van Kampen theorem, we see that X is simply-connected. Using the Mayer-Vietoris homology exact sequence, we see that the homology groups of X are isomorphic to the homology groups of the product space, $S^3 \times S^5$.

Finally, we must justify calling X an 8-dimensional *manifold*, by showing that X is locally homeomorphic to the Euclidean space, $\mathbb{R}^8$. The points of X are represented by points of the cylinder, $W \times [0,1] = \{\, (w,t) \mid w \in W,\ 0 \leq t \leq 1 \,\}$. For points (w,t) with $0 < t < 1$, the existence of a neighborhood homeomorphic to $\mathbb{R}^8$ follows at once from the fact that W is a 7-dimensional manifold.

For those points of X which are equivalence classes of more than one element, they are represented by either $(w,0)$ or $(w,1)$, for some $w \in W$. It is easy to construct a *neighborhood* of such a point in X as the *product* of *two spaces*; the first is a sufficiently small neighborhood of the point in either S^5 or M^5 (and thus, may be chosen to be homeomorphic to $\mathbb{R}^5$), while the second space is the *cone on the fibre* of the appropriate attaching map (since it is a fiber bundle projection). Of course, p and q each have fibres homeomorphic to S^2, and so the second space can be viewed as a cone homeomorphic to $\mathbb{R}^3$. Therefore, we have constructed neighborhoods of points represented by pairs (w,t) with $t = 0$ or 1, and these neighborhoods are homeomorphic to $\mathbb{R}^5 \oplus \mathbb{R}^3 \cong \mathbb{R}^8$. We conclude that X is locally homeomorphic to the Euclidean space, $\mathbb{R}^8$, near each of its points.

To summarize, we have the following.

Proposition: X is a compact, simply-connected, 8-dimensional manifold, and

$$H_p(X) \cong H_p(S^3 \times S^5)$$

are isomorphic integral homology groups, for all $p \geq 0$. ∎

From these facts it follows that the *homology decomposition* [7] for the homotopy type of X is given by

$$X = (S^3 \cup_\alpha e^5) \cup_\beta e^8,$$

where $\alpha \in \pi_4(S^3) = Z_2$, $\beta \in \pi_7(S^3 \cup_\alpha e^5)$ and X is homotopy equivalent to the cofibre (or mapping cone) of β. Of course, the homotopy class $\alpha = 0$ if and only if $\mathrm{Sq}^2 : H^3(X^8; Z_2) \to H^5(X^8; Z_2)$ is zero, and this occurs, if and only if $\mathrm{Sq}^2 : H^3(M^5; Z_2) \to H^5(M^5; Z_2)$ is zero.

To see that $\mathrm{Sq}^2 : H^3(M^5; Z_2) \to H^5(M^5; Z_2)$ is zero, consider the appropriate non-zero transgression in the Serre cohomology spectral sequence with Z_2 coefficients for the principal $SO(3)$-bundle with projection $SU(3) \to M^5$.

Therefore, $\alpha = 0$ and $\beta \in \pi_7(S^3 \vee S^5) \cong \pi_7(S^3) \oplus \pi_7(S^5) \oplus \pi_8(S^8)$. Though tangential to the main thrust of this paper, it may be interesting to ask for which elements $\beta \in \pi_7(S^3 \vee S^5)$ does the cofibre of β have the homotopy type of a manifold?

We can extend the previous construction to larger square matrices. However, unlike the series of examples produced in the next section, attempts to generalize the above example lead to the construction of spaces which are not manifolds, since they fail to be locally Euclidean.

The Lie group $G = SU(n)$ contains the two closed subgroups, $G_0 = SU(n-1)$ and $G_1 = SO(n)$, where the inclusion of $SU(n-1)$ in $SU(n)$ uses the upper left $(n-1) \times (n-1)$ submatrix of the $(n \times n)$ matrix, while $SO(n)$ is the subgroup consisting of those matrices of $SU(n)$ containing only real entries, and therefore, the intersection is $SU(n-1) \cap SO(n) = SO(n-1)$.

Now consider the manifold, $W_{(n)} = SU(n)/SO(n-1)$ and view the two attaching maps as the canonical fibre bundle projections,

$$p : W_{(n)} \to S^{2n-1} = SU(n)/SU(n-1)$$

and

$$q : W_{(n)} \to M_{(n)} = SU(n)/SO(n).$$

Each of these fibre bundles has its fibres homeomorphic to a manifold. The fibre of $q : W_{(n)} \to M_{(n)}$ is just the sphere, S^{n-1}. However, the fibre of $p : W_{(n)} \to S^{2n-1}$ is $M_{(n-1)}$, which is certainly not an integral homology sphere, for each $n \geq 4$. This follows from the fact that the integral homology group, $H_2(M_{(n-1)})$ is the *group of order two*, in case $n \geq 4$. (The case $n = 2$ is trivial, while the case $n = 3$ is discussed at length in the earlier part of this section.)

Now we can define $X_{(n)}$ to be the double mapping cylinder

$$X_{(n)} = S^{2n-1} \cup_p \left(W_{(n)} \times [0,1] \right) \cup_q M_{(n)},$$

where again the cylinder on $W_{(n)}$ is attached at each end to the base space of the indicated fibre bundle projection. From the Van Kampen theorem, we see that $X_{(n)}$ is simply-connected. Of course, the homology of $X_{(n)}$ can be computed using the Mayer-Vietoris homology exact sequence.

Finally, we should examine $X_{(n)}$ to try to understand why it is *not a manifold*, for $n \geq 4$. There are three mutually disjoint subsets of $X_{(n)}$, whose union is $X_{(n)}$. One is the open cylinder in the middle, $W_{(n)} \times (0,1)$, and the other two are the closed subsets, $M_{(n)}$ and S^{2n-1}. Obviously, the points of $W_{(n)} \times (0,1)$ are non-singular. Further, the points of $M_{(n)} \subset X_{(n)}$ are seen to have Euclidean neighborhoods, which can each be constructed as a product of a Euclidean neighborhood in $M_{(n)}$ and an open cone on S^{n-1}, the fibre of the projection, q. The third set, S^{2n-1} is quite different.

The points of the embedded sphere, $S^{2n-1} \subset X_{(n)}$ are precisely the *singular points* of $X_{(n)}$ (i.e., those points of $X_{(n)}$ which do not possess a neighborhood homeomorphic to a Euclidean space). However, neighborhoods of these singular points can be constructed as products of open cones on spaces. Thus, these neighborhoods are themselves open cones on the joins of the corresponding spaces. Specifically, for a point in S^{2n-1}, a neighborhood in $X_{(n)}$ can be constructed as a product of a small neighborhood in S^{2n-1} and an open cone on $M_{(n-1)}$, the fibre of the projection, p. Since $M_{(n-1)}$ is not an homology sphere, for $n \geq 4$, the local homology at the vertex of the cone shows that the vertex does *not* have a neighborhood homeomorphic to a Euclidean space. Details on the construction and analysis of these locally contractible spaces in which all points have *conical neighborhoods*, will be found in [5, Appendix D]. Of course, we can regard these neighborhoods as generalized polar local coordinates.

4. The $(4n+1)$-dimensional manifold, Y^{4n+1}

For each integer, $n \geq 2$, the Lie group $G = SU(n+1)$ contains the two closed subgroups, $G_0 \cong SU(n)$ and $G_1 \cong SU(n)$, where the two inclusions of $SU(n)$ in $SU(n+1)$ make use of the usual upper left and lower right $(n \times n)$ submatrices of the $\Big((n+1) \times (n+1)\Big)$ matrix. Of course, $G_0 \cap G_1 = SU(n-1)$.

In this section we apply the canonical construction (described in section 1) to this lattice of subgroups. Here again, since the lattice has only the two subgroups and their intersection, the construction yields

$$Y = (G/G_0) \cup_p \Big((G/(G_0 \cap G_1)) \times [0,1]\Big) \cup_q (G/G_1)$$

Let $W = W_{n+1,2} = SU(n+1)/SU(n-1)$ denote the $4n$-dimensional complex Stiefel manifold of orthonormal complex 2-frames in C^{n+1}. The two attaching maps, p and q (at the two ends of the cylinder) are the canonical fibre bundle projections,

$$p, q : W \to S^{2n+1} = SU(n+1)/SU(n).$$

Both of these fibre bundles have fibres homeomorphic to $SU(n)/SU(n-1) = S^{2n-1}$, the $(2n-1)$-dimensional sphere. Therefore, W is a simply-connected, $(4n)$-dimensional manifold with homology isomorphic to the homology of the product space, $S^{2n-1} \times S^{2n+1}$.

Now we can define the $(4n+1)$-dimensional manifold Y as the following double mapping cylinder.

$$Y^{4n+1} = S^{2n+1} \cup_p \Big(W^{4n} \times [0,1]\Big) \cup_q S^{2n+1},$$

where the cylinder on W is attached at each end, to the base space of the indicated fibre bundle projection. Again, the Van Kampen theorem, implies that Y is simply-connected, and the Mayer-Vietoris homology exact sequence implies that the homology of Y is isomorphic to the homology of the product space, $S^{2n} \times S^{2n+1}$.

Finally, we show that Y is a $(4n+1)$-dimensional *manifold*, by proving that Y is locally homeomorphic to the Euclidean space, $\mathbb{R}^{4n+1}$. As in the last section, the points of Y are represented by points of the cylinder, $W \times [0,1] = \{\, (w,t) \mid w \in W,\ 0 \leq t \leq 1 \,\}$. For points (w,t) with $0 < t < 1$, the existence of a neighborhood homeomorphic to $\mathbb{R}^{4n+1}$ follows at once from the fact that W is a $(4n)$-dimensional manifold.

For those points of Y which are equivalence classes of more than one element, they are represented by either $(w,0)$ or $(w,1)$, for some $w \in W$. Again, it is possible to construct a *neighborhood* of such a point in Y (at either end), as the *product* of *two spaces*. The first of these spaces is any sufficiently small neighborhood of the point in S^{2n+1} (and thus, may be chosen to be homeomorphic to $\mathbb{R}^{2n+1}$), while the second of these spaces is the *cone on the fibre* of the appropriate attaching map (and fiber bundle projection). Of course, in both cases, p and q each have fibres homeomorphic to S^{2n-1}, and so the second space can be viewed as a cone homeomorphic to $\mathbb{R}^{2n}$. Therefore, we have constructed neighborhoods (of points represented by (w,t) with $t = 0$ or 1), which are homeomorphic to $\mathbb{R}^{2n+1} \oplus \mathbb{R}^{2n} \cong \mathbb{R}^{4n+1}$. We conclude that Y is *locally homeomorphic* to the Euclidean space, $\mathbb{R}^{4n+1}$, near each of its points. In summary, we have the following.

Proposition: If integer $n \geq 2$, then Y is a compact, simply-connected, $(4n+1)$-dimensional manifold, and

$$H_p(Y) \cong H_p(S^{2n} \times S^{2n+1})$$

are isomorphic integral homology groups, for all $p \geq 0$. ∎

Much of the remainder of this section is an explanation of the following result. The author wishes to thank A. Borel [1] for the insightful suggestion that it seemed rather likely that Y^{4n+1} could be identified in this way.

Proposition: If integer $n \geq 2$, then Y^{4n+1} is homeomorphic to $V_{2n+2,2}$, the unit tangent bundle of S^{2n+1} (i.e., the real Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{2n+2}$). ∎

To prove that $Y^{4n+1} \approx V_{2n+2,2}$, we will first construct an action of $SU(n+1)$ on $V_{2n+2,2}$ and then identify the orbit space of this action with the closed interval, $[-1,1] \subset \mathbb{R}$. Then it remains only to show that the isotropy subgroup of $SU(n+1)$ for an orbit is $SU(n)$ (respectively, $SU(n-1)$), if the orbit is represented as an endpoint (respectively, interior point) of $[-1,1]$, the orbit space.

Explicitly, using column vectors, let $z = (z_1, \ldots, z_{n+1})^T$, $w = (w_1, \ldots, w_{n+1})^T \in \mathbb{C}^{n+1}$, where each $z_k = a_k + \imath b_k$ and $a_k, b_k \in \mathbb{R}$. Consider the $\mathbb{R}$-module isomorphism, $\widehat{\ } : \mathbb{C}^{n+1} \to \mathbb{R}^{2n+2}$ which sends z to $\widehat{z} = x = (a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1})^T$ (which defines x). Similarly, define $y = \widehat{w}$. Suppose $(x, y) \in \mathbb{R}^{2n+2} \times \mathbb{R}^{2n+2}$ is an orthonormal pair. Therefore, $z^*z = x^Tx = 1 = y^Ty = w^*w$ and $x^Ty = 0$. Further, the usual complex inner product on $\mathbb{C}^{n+1}$ is defined by z^*w. Since the real part of z^*w is just $x^Ty = 0$, it follows that $\imath z^*w \in \mathbb{R}$. In fact, $-1 \leq \imath z^*w \leq 1$, and $\imath z^*w$ is the required invariant parameter which characterizes orbits. This identifies the orbit space as $[-1,1]$.

Notice that $\{z, w\}$ is linearly dependent (respectively, independent) over $\mathbb{C}$, if and only if $\imath z^*w = \pm 1$ (respectively, $-1 < \imath z^*w < 1$), and this occurs if and only if the isotropy subgroup, $H = SU(n)$ (respectively, $H = SU(n-1)$). Thus, we have a representation of $V_{2n+2,2}$ as the *double mapping cylinder*

$$V_{2n+2,2} = S^{2n+1} \cup_\phi \left(W_{n+1,2} \times [-1,1] \right) \cup_\psi S^{2n+1},$$

which is the definition of Y^{4n+1}.

Finally, $V_{2n+2,2}$ can be viewed as the total space of the unit tangent vector bundle over S^{2n+1}, consisting of all the unit length tangent vectors for $S^{2n+1} \subset \mathbb{R}^{2n+2}$. Each fibre is homeomorphic to S^{2n} and contains two distinguished points (a copy of S^0), corresponding (under the $\widehat{\ }$ identification) to the scalar multiplication by $\pm i$ of the unit vector under this fibre in the base, $S^{2n+1} \subset \mathbb{C}^{n+1}$. In each fibre, the isotropy subgroup is $SU(n)$ on S^0, and $SU(n-1)$ on the complement, $S^{2n} \setminus S^0$. The union of all these copies of S^0 is the total space of a subbundle with fibre S^0 and base S^{2n+1}. Since the base is simply connected, this covering space is a product, $S^0 \times S^{2n+1} \subset V_{2n+2,2}$, and consists of the two ends of the double mapping cylinder. These two copies of S^{2n+1} are the two non-generic orbits, and they represent the standard example of a unit tangent vector field (and its negative) on an odd dimensional sphere. Their complement in $V_{2n+2,2}$ is then the cylinder $W_{n+1,2} \times (-1,1)$, which projects to the base S^{2n+1} with fibre $S^{2n-1} \times (-1,1)$. Thus, the construction (over S^{2n+1}) of $V_{2n+2,2}$ from $W_{n+1,2}$ appears as a *suspension* of the fibres.

5. Generalizations and Conclusions

The *lattice construction* occurs quite naturally in the *adjoint representation* of a Lie group on its Lie algebra. Certain special cases are studied in detail in [5] [3] [4] (with an early report in [6]), with the two primary purposes of *explicitly computing* (1) the spectra of symmetric molecules with a strong *Jahn-Teller effect*, and (2) the *quantum adiabatic phase* (or "Berry phase" or "quantum geometric phase") of all possible, *time-reversal invariant* quantum systems, in terms of simple algebraic topology invariants of the given periodic Hamiltonian. The solution of this general "Berry phase" problem requires an understanding of the stratification provided by the lattice construction for the following lattices of closed subgroups of two ambient Lie groups, G. For each integer, $n \geq 2$, the ambient group, $G = U(n)$ (respectively, $G = O(n)$) contains the lattice of all possible intersections of the $n-1$ subgroups $\{ U(k) \times U(n-k) \mid k = 1, 2, \ldots, n-1 \}$ (respectively, $\{ O(k) \times O(n-k) \mid k = 1, 2, \ldots, n-1 \}$). Regarding molecular spectra, the Jahn-Teller effect is seen to be the result of the Riemannian geometry of this lattice construction (using the orthogonal groups) for $2 \leq n \leq 5$.

Of course, a similar construction can be obtained using the symplectic groups and the lattices of block-diagonal products of smaller symplectic groups. This symplectic group construction is not discussed in [5], since it is not required in our treatment of the Jahn-Teller effect and the quantum adiabatic phase.

Returning to the notation introduced in section 1, $m = n - 2$ and the quotient space, $X = (G \times \Delta_m)/(\sim)$ is diffeomorphic to a sphere of dimension $n^2 - 2$ (respectively, a sphere of dimension $(n^2 + n - 4)/2$). Thus, section 2 of this paper is just the lattice construction for $n = 4$ with $G = U(4)$, together with the observation that the quotient space, X is the 14-sphere. In section 2 it is clear, a priori (by construction), that $X = S^{14}$, and so (unlike the examples in sections 3 and 4), it is not necessary to prove that X is locally Euclidean. However, it is an enlightening exercise to show (directly from the lattice construction) that these quotient spaces, X, are locally Euclidean. This exercise is nearly trivial for $n \leq 3$ but for $n = 4$ it is non-trivial to show that X is locally homeomorphic to $\mathbb{R}^{14}$ (without using the fact that $X = S^{14}$, which we happen to know, only because we understand that the construction of X is equivalent to the stratification of S^{14} obtained from the decomposition [1] into orbit types, using the given $U(4)$ action).

Note that from the unitary group, $U(n)$, we construct the map

$$\beta : S^{n^2-2} \to \Delta_{n-2},$$

while the orthogonal group, $O(n)$, yields the map

$$\beta : S^{(n^2+n-4)/2} \to \Delta_{n-2}.$$

Both of these maps are denoted by the same symbol, β, because the latter is just the restriction of the former. In these examples, the map β is merely the global geometric manifestation of the finite dimensional spectral theorem.

Thus far, the lattice construction has always produced surjective maps

$$\beta : X \to \Delta_m$$

which have a standard simplex as the range space. In order to generalize [2] this construction, consider an arbitrary finite simplicial complex, K. Suppose that K^0, the zero skeleton, consists of $m + 1$ vertices. Then

[1] Dissection is easier than construction, as with frogs.
[2] For a somewhat different generalization, see [5, Appendix D].

K is a subcomplex of Δ_m. Suppose further that $\beta : X \to \Delta_m$ is the result of some lattice construction. This permits the definition of $X(K) = \beta^{-1}(K)$, and the restriction of β produces the map

$$\beta_K : X(K) \to K,$$

which can be viewed as the pull-back of β over the inclusion, $K \subset \Delta_m$. This map, β_K is called the *lattice construction over the simplicial complex,* K and it is the desired generalization. Obviously, the space, $X(K)$ may depend strongly on the chosen *bijective correspondence* between the set of $m+1$ closed subgroups of G, and the set of $m+1$ vertices that make up the zero skeleton, K^0.

Finally, in this somewhat greater generality, we return to the *skeletally induced filtration* introduced in section 1. Let K^p denote the p-skeleton of the finite, simplicial complex, K, and *define* the finite filtration on $X(K)$ given by

$$X_p = X(K^p) = {\beta_K}^{-1}(K^p)$$
$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_m = X(K)$$

As usual, this filtration gives rise to various spectral sequences. This paper concludes with a brief outline of the corresponding spectral sequence for ordinary cohomology (with an arbitrary abelian group of coefficients, which is not displayed in the notation). Of course, any general homology or cohomology theory would yield a similar spectral sequence.

Consider the exact sequence for the pair, (X_p, X_{p-1}).

$$\cdots \longrightarrow H^{p+q-1}(X_{p-1}) \longrightarrow H^{p+q}(X_p, X_{p-1}) \longrightarrow H^{p+q}(X_p) \longrightarrow H^{p+q}(X_{p-1}) \longrightarrow \cdots$$

Definition: Let $E_1^{p,q} = H^{p+q}(X_p, X_{p-1})$ and let $D_1^{p,q} = H^{p+q}(X_{p-1})$ ∎

Thus, the exact sequence for this pair can be written

$$\cdots \longrightarrow D_1^{p,q-1} \longrightarrow E_1^{p,q} \longrightarrow D_1^{p+1,q-1} \longrightarrow D_1^{p,q} \longrightarrow \cdots$$

for all integers, p. This *exact couple* gives rise to the associated spectral sequence in the usual way. (See [8] and [11] for a description of this technology.)

Definition: Let $A^{p,q}$ denote the kernel of the restriction $H^{p+q}(X(K)) \to H^{p+q}(X_{p-1})$. ∎

This produces the filtration of $H^{p+q}(X(K))$,

$$0 = A^{m+1,p+q-m-1} \subset \cdots \subset A^{p+1,q-1} \subset A^{p,q} \subset \cdots \subset A^{0,p+q} = H^{p+q}(X(K))$$

and the usual exact couple argument proves that

$$0 \longrightarrow A^{p+1,q-1} \hookrightarrow A^{p,q} \hookrightarrow E_\infty^{p,q} \longrightarrow 0$$

is a *short exact sequence*. Thus, this spectral sequence converges to the graded object associated with the $A^{*,*}$-filtration of $H^*(X(K))$, as one might expect from such an exact couple.

As observed in the introduction, the only unusual feature of this spectral sequence is the passage from the $E_1^{*,*}$ terms to the $E_2^{*,*}$ terms. Of course, in general we have differentials on level s.

$$d_s : E_s^{p,q} \to E_s^{p+s,q-s+1},$$

We can give an explicit description in one particular case, the differentials on level 1,

$$d_1 : E_1^{p,q} = H^{p+q}(X_p, X_{p-1}) \to H^{p+q+1}(X_{p+1}, X_p) = E_1^{p+1,q}$$

because of the natural direct sum decomposition of the E_1 terms,

$${E_1}^{p,q} = H^{p+q}(X_p, X_{p-1}) \cong \sum_{\sigma} H^q(G/G[\sigma]),$$

where σ indexes the p-simplices of K and

$$G[\sigma] = \bigcap_{i=0}^{p} G_{r_i},$$

$\{b_{r_0}, b_{r_1}, \ldots, b_{r_p}\}$ being the non-zero barycentric coordinates in the interior of the p-cell σ

The non-zero components of d_1 occur for pairs, (τ, σ) where σ is a p-simplex in the boundary of the $(p+1)$-simplex $\tau \subset K$. Let b_q be the additional non-zero barycentric coordinate in the interior of the $(p+1)$-cell τ. Thus, we have $G[\tau] = G_q \cap G[\sigma]$, and the natural fibre bundle projection $\Phi_{\tau,\sigma} : G/G[\tau] \to G/G[\sigma]$ has fibre homeomorphic to $G[\sigma]/G[\tau] = G[\sigma]/(G_q \cap G[\sigma])$, and induces the corresponding component of d_1 on the cohomology summands

$${\Phi_{\tau,\sigma}}^* \ : \ H^*(G/G[\sigma]) \to H^*(G/G[\tau]).$$

Thus, the components of d_1 can be calculated by viewing them (each) as *edge morphisms* in a Serre cohomology spectral sequence for the fibration, $\Phi_{\tau,\sigma}$.

Notice that all the examples of sections 3 and 4 are concerned with $X(\Delta_1)$, the case in which $K = \Delta_1$. The spectral sequence under discussion simplifies considerably when $K = \Delta_1$ becoming merely a *long exact sequence*[3]. It is easily seen that this *long exact sequence* is naturally equivalent to the *Mayer-Vietoris exact sequence*, which proved useful in sections 3 and 4.

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