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Eigenvalue Asymptotics for the Dirac Operator in Strong Constant Magnetic Fields

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Abstract. We consider the three-dimensional Dirac operator H with constant magnetic field and electric potential which decays at infinity. We study the asymptotic behaviour of the discrete spectrum of H as the norm of the magnetic field grows unboundedly.

1 Introduction

Let $H_0(b)$ be the three-dimensional Dirac operator in constant magnetic field $B = (0,0,b)$, $b>0$. Choosing an appropriate gauge, system of units, and coordinates, we can write

$$H_0(b) = \sum_{j=1,2,3} \alpha_j \Pi_j(b) + \beta$$

where

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix},\ j=1,2,3,\ \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

σ_j, $j=1,2,3$, are the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

I_2 is the unit 2×2 matrix, Π_j, $j = 1,2,3$, are the components of the extended momentum

$$\Pi_1 = \Pi_1(b) := -i\frac{\partial}{\partial x} + \frac{by}{2}, \Pi_2 = \Pi_2(b) := -i\frac{\partial}{\partial y} - \frac{bx}{2}, \Pi_3 := -i\frac{\partial}{\partial z},$$

and $X = (x,y,z) \in \mathbf{R}^3$. It is well-known that for each $b \geq 0$ we have

$$\sigma(H_0(b)) = \sigma_{\rm ess}(H_0(b)) = (-\infty,-1] \cup [1,+\infty). \tag{1.1}$$

Further, let $V : \mathbf{R}^3 \to \mathbf{R}$ be the electric (scalar) potential. We shall say that V is in the class $\mathcal{L}$ if and only if for each $\varepsilon > 0$ it can be written as $V = V_1 + V_2$ with $V_1 \in L^3(\mathbf{R}^3)$, and $\sup_{X\in\mathbf{R}^3} |V_2(X)| \leq \varepsilon$. Throughout the paper we assume $V \in \mathcal{L}$, unless more restrictive assumptions are imposed.
In particular, $V \in \mathcal{L}$ entails the compactness of the operator $VH_0(b)^{-1}$. Set

$$H(b) := H_0(b) + VI_4 = H_0(b) + V$$

where I_4 is the unit 4×4 matrix. Since the operator $VH_0(b)^{-1}$ is compact, we have $\sigma_{\rm ess}(H(b)) = \sigma_{\rm ess}(H_0(b))$, and hence (1.1) implies

$$\sigma_{\rm ess}(H(b)) = (-\infty,-1] \cup [1,+\infty).$$

However, the discrete spectrum of the operator $H(b)$ might be non-empty. The aim of the present paper is to investigate the asymptotic distribution as $b\to\infty$ of the eigenvalues of $H(b)$ lying in the gap $(-1,1)$ of its essential spectrum.

2 Statement of the main result

Let $T = T^*$ be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbf{R}$. Set

$$\mathcal{N}(\lambda_1, \lambda_2; T) = \text{rank}\, P_{(\lambda_1,\lambda_2)}(T),\ \lambda_1, \lambda_2 \in \mathbf{R},\ \lambda_1 < \lambda_2,$$

$$N(\lambda; T) := \text{rank}\, P_{(-\infty,\lambda)}(T),\ \lambda \in \mathbf{R},$$

$$n_{\pm}(s; T) := \text{rank}\, P_{(s,+\infty)}(\pm T), s > 0.$$

If T is a linear compact operator which is not necessarily selfadjoint, put

$$n_*(s; T) := \text{rank}\, P_{(s^2,+\infty)}(T^*T), s > 0.$$

In what follows if $X = (x, y, z) \in \mathbf{R}^3$ we shall write occasionally $X = (X_\perp, z)$ where $X_\perp = (x, y)$ are the variables on the plane perpendicular to the magnetic field $B = (0, 0, b)$, while z is the variable along B. Fix $X_\perp \in \mathbf{R}^2$ and set

$$\chi(X_\perp) := \chi_0 + V(X_\perp, .)\, I_2$$

where

$$\chi_0 := \begin{pmatrix} 1 & -i\frac{d}{dz} \\ -i\frac{d}{dz} & -1 \end{pmatrix}.$$

Proposition 2.1 *Let $V \in \mathcal{L}$. Then for almost every $X_\perp \in \mathbf{R}^2$ the operator $\chi(X_\perp)$ is defined as an operator sum selfadjoint in $L^2(\mathbf{R}; \mathbf{C}^2)$. Moreover, for almost every $X_\perp \in \mathbf{R}^2$ the operator $V(X_\perp, .)\chi_0^{-1}$ is compact and, therefore,*

$$\sigma_{\text{ess}}(\chi(X_\perp)) = \sigma_{\text{ess}}(\chi_0) = (-\infty, -1] \cup [1, +\infty).$$

The proof of the proposition is contained in Section 7.
Let λ_1 and λ_2 be real numbers such that $-1 < \lambda_1 < \lambda_2 < 1$. Introduce the magnetic integrated density of states

$$\mathcal{D}(\lambda_1, \lambda_2) = \mathcal{D}_V(\lambda_1, \lambda_2) := \int_{\mathbf{R}^2} \mathcal{N}(\lambda_1, \lambda_2; \chi(X_\perp))\, dX_\perp.$$

Proposition 2.2 *Let $V \in \mathcal{L}$, $\lambda_1, \lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Then $\mathcal{D}_V(\lambda_1, \lambda_2) < \infty$.*

The proof of this proposition can also be found in Section 7.
We shall say that a point $\lambda \in (-1, 1)$ is regular if and only if

$$\text{vol}\, \left\{X_\perp \in \mathbf{R}^2 | \dim \text{Ker}\, (\chi(X_\perp) - \lambda) \ge 1\right\} = 0.$$

Note that λ_1 (respectively, λ_2) is a regular point if and only if $\lim_{\varepsilon\to 0} \mathcal{D}(\lambda_1 + \varepsilon, \lambda_2) = \mathcal{D}(\lambda_1, \lambda_2)$ (respectively, $\lim_{\varepsilon\to 0} \mathcal{D}(\lambda_1, \lambda_2 + \varepsilon) = \mathcal{D}(\lambda_1, \lambda_2)$).

Theorem 2.1 *Let $V \in \mathcal{L}$, $\lambda_1, \lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Assume that the points λ_1 and λ_2 are regular. Then we have*

$$\lim_{b \to \infty} b^{-1} \mathcal{N}(\lambda_1, \lambda_2; H(b)) = \frac{1}{2\pi} \mathcal{D}(\lambda_1, \lambda_2). \tag{2.1}$$

The present paper could be regarded as a supplement to [7] where strong-magnetic-field spectral asymptotics for the Schrödinger and Pauli operators have been considered. The methods applied here are close to the ones used in [7]. However, the Dirac operator $H(b)$ studied in this paper as well as the auxiliary operator $\chi(X_\perp)$ are not semibounded in contrast to the Schrödinger and Pauli operators. This additional difficulty is overcome by the application of a simple but yet non-trivial generalization of the well-known Birman-Schwinger principle.
Various types of spectral properties and, in particular, eigenvalue asymptotics for the Dirac operator with or without magnetic field have been studied in [8], [3], [4], [5], [6]. However, the asymptotic behaviour as $b \to \infty$ of $\mathcal{N}(\lambda_1, \lambda_2; H(b))$ has never been investigated before.
The paper is organized as follows. The next four brief sections contain auxiliary results. A formulation of the Kac-Murdock-Szegö theorem borrowed from [7], can be founded in Section 3. Section 4 is devoted to the generalization of the Birman-Schwinger principle concerning the number of the eigenvalues situated in a gap of the essential spectrum of a selfadjoint operator. In Section 5 we describe certain spectral properties of the unperturbed operator $H_0(b)$. In Section 6 we perform some preliminary estimates. Finally, Propositions 2.1-2.2 are proved in Section 7, and Theorem 2.1 – in Section 8.

3 The Kac-Murdock-Szegö theorem

In this section we follow closely the exposition of [7, Subsection 3.1]. For the reader's convenience, we reproduce a suitable version of the Kac-Murdock-Szegö theorem whose proof can be found in [7, Subsection 3.1].
In the sequel we shall denote by S_∞ the space of linear compact operators acting in a given Hilbert space, and by S_p, $p \in [1, \infty)$, – the Schatten–von Neumann spaces of operators $T \in S_\infty$ for which the norm $\|T\|_p := (\mathrm{Tr}\,|T|^p)^{1/p}$ is finite.
Moreover, we shall say that the function ν defined on $\mathbf{R} \setminus \{0\}$ is in the class $\mathcal{C}$ if it is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$, non-negative on $(-\infty, 0)$, and non-positive on $(0, \infty)$.

Lemma 3.1 *Let $\{T(b)\}_{b>0}$ be a family of selfadjoint compact operators satisfying the estimate $\|T(b)\| \le t_0$ with $t_0 > 0$ independent of b. Let $\nu \in \mathcal{C}$. Assume that $\nu(t) = 0$ for $|t| > t_0$. Suppose that there exists a real $p \ge 1$ such that the following three conditions are fulfilled*:
(i) $T(b) \in S_p$ *for each* $b > 0$;

(ii) *the quantity* $\int_{\mathbf{R}\setminus\{0\}} |t|^p \, d\nu(t)$ *is finite;*
(iii) *the limiting relations*

$$\lim_{b\to\infty} b^{-1} \text{ Tr } T(b)^l = \int_{\mathbf{R}\setminus\{0\}} t^l \, d\nu(t)$$

hold for each integer $l \geq p$.
Let $t \neq 0$ *be a continuity point of* ν. *Then we have*

$$\lim_{b\to\infty} b^{-1} \, n_-(-t; T(b)) = \nu(t) \quad \text{if} \quad t < 0,$$

$$\lim_{b\to\infty} b^{-1} \, n_+(t; T(b)) = -\nu(t) \quad \text{if} \quad t > 0.$$

Remark. We shall use Lemma 3.1 only with $t < 0$.

4 The generalized Birman-Schwinger principle

One of the versions of the classical Birman-Schwinger principle (cf. [2, Lemma 1.1]) says that if $\mathcal{H}_0 = \mathcal{H}_0^* \geq 0$, $\mathcal{V} = \mathcal{V}^*$, and $|\mathcal{V}|^{1/2}(\mathcal{H}_0 + 1)^{-1/2} \in S_\infty$, then for each $\lambda > 0$ we have

$$N(-\lambda; \mathcal{H}_0 + \mathcal{V}) = n_-(1; (\mathcal{H}_0 + \lambda)^{-1/2}\mathcal{V}(\mathcal{H}_0 + \lambda)^{-1/2}) \tag{4.1}$$

where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the quadratic-forms sense.
Lemma 4.1 below contains a generalization of (4.1) to the case where $\mathcal{H}_0$ is not necessarily semibounded.
Related arguments in the special case where $\mathcal{H}_0$ coincides with the free Dirac operator have already appeared in [2, Section 5]. Much later arguments of this type have been employed in [4] and [5].
Let $\mathcal{H}_0$ be a linear operator selfadjoint in the Hilbert space H. Assume $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 < \lambda_2$, $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0)$.
Set $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) := ((\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2))^{-1/2}$. Since $[\lambda_1, \lambda_2]$ is in the resolvent set of $\mathcal{H}_0$ the operator $(\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2)$ is positive-definite, and hence the operator $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)$ is well-defined and bounded. Set

$$\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0) := \left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right) \mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0).$$

Evidently, $\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)$ is bounded. Further, let $\mathcal{V}$ be a symmetric operator on $D(\mathcal{H}_0)$ such that $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in S_\infty$, which is equivalent to $\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) \in S_\infty$. Set

$$\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) :=$$

$$\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}^2\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) + 2\text{Re}\, \mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0).$$

Obviously, $\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) \in S_\infty$.

Lemma 4.1 *Let $\mathcal{H}_0$ be a linear operator selfadjoint in the Hilbert space* H, $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 < \lambda_2$, *and* $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0)$. *Let $\mathcal{V}$ be a symmetric operator on $D(\mathcal{H}_0)$ such that* $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in S_\infty$. *Then we have*

$$\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})) \tag{4.2}$$

where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the operator sense.

Proof. Obviously,

$$\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = N\left(\frac{1}{4}(\lambda_1 - \lambda_2)^2; (\mathcal{H}_0 + \mathcal{V} - \frac{1}{2}(\lambda_1 + \lambda_2))^2\right). \tag{4.3}$$

The minimax principle implies that the quantity at the right-hand side of (4.3) is equal to the maximal dimension of the linear subsets of $D(\mathcal{H}_0)$ whose non-zero elements u satisfy the inequality

$$\left\|\mathcal{H}_0 u + \mathcal{V}u - \frac{1}{2}(\lambda_1 + \lambda_2)u\right\|^2 < \frac{1}{4}(\lambda_1 - \lambda_2)^2\|u\|^2$$

where $\|.\|$ denotes the norm in H. This last inequality can be re-written as

$$\|\sqrt{(\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2)}u\|^2 < -\|\mathcal{V}u\|^2 - 2\mathrm{Re}\left\langle\left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right)u, \mathcal{V}u\right\rangle \tag{4.4}$$

where $\langle .,. \rangle$ denotes the inner product in H. Note that the operator $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)$ maps bijectively H on $D(\mathcal{H}_0)$. Set $u = \mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w$, $w \in$ H, in (4.4). Hence, (4.4) is equivalent to

$$\begin{aligned} \|w\|^2 < -\|\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w\|^2 - \\ 2\mathrm{Re}\left\langle\left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right)\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w, \mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w\right\rangle = \\ -\left\langle \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})w, w\right\rangle. \end{aligned} \tag{4.5}$$

By (4.3), the quantity $\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V})$ coincides with the maximal dimension of the subspaces of H whose non-zero elements w satisfy (4.5). By the minimax principle this maximal dimension equals $n_-(1; \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}))$. Hence, (4.2) is valid. □

In this paper we shall apply Lemma 4.1 in the case $\mathcal{H}_0 = H_0(b)$, $\mathcal{V} = V$, and $[\lambda_1, \lambda_2] \subset (-1, 1)$. Set

$$R(\lambda_1, \lambda_2) \equiv R_b(\lambda_1, \lambda_2) := \mathcal{R}(\lambda_1, \lambda_2; H_0(b)), \tag{4.6}$$

$$G(\lambda_1, \lambda_2) \equiv G_b(\lambda_1, \lambda_2) := \mathcal{G}(\lambda_1, \lambda_2; H_0(b)), \tag{4.7}$$

$$K(\lambda_1, \lambda_2) \equiv K_b(\lambda_1, \lambda_2) := \mathcal{K}(\lambda_1, \lambda_2; H_0(b), V). \tag{4.8}$$

By analogy with (4.6) and (4.7) introduce the operators $\varrho(\lambda_1, \lambda_2)$ and $\gamma(\lambda_1, \lambda_2)$ replacing $H_0(b)$ by χ_0. Similarly, fix $X_\perp \in \mathbf{R}^2$ such that the operator $\chi(X_\perp)$ is well-defined,

and $V(X_\perp,.)\chi_0^{-1} \in S_\infty$, and define the operator $\kappa(X_\perp) \equiv \kappa(\lambda_1,\lambda_2;X_\perp)$ substituting in (4.8) the operator $H_0(b)$ for χ_0, and V for $V(X_\perp,.)$.
Applying (4.2), we obtain

$$\mathcal{N}(\lambda_1,\lambda_2;H(b)) = n_-(1;K_b(\lambda_1,\lambda_2)), \tag{4.9}$$

$$\mathcal{D}(\lambda_1,\lambda_2) = \int_{\mathbf{R}^2} n_-(1;\kappa(\lambda_1,\lambda_2;X_\perp))\,dX_\perp. \tag{4.10}$$

For further references we formulate here a lemma which is closely related to the generalized Birman–Schwinger principle.

Lemma 4.2 *Let the operators $\mathcal{H}_0$ and $\mathcal{V}$ and the numbers $\lambda_1,\ \lambda_2 \in \mathbf{R}$ satisfy the hypotheses of Lemma* 4.1. *Then the spectrum of $\mathcal{H}_0+\mathcal{V}$ contains at least one of the points λ_1 and λ_2 if and only if the operator $\mathcal{K}(\lambda_1,\lambda_2;\mathcal{H}_0,\mathcal{V})$ has an eigenvalue equal to -1. Moreover,*

$$\sum_{j=1,2} \dim \operatorname{Ker}(\mathcal{H}_0+\mathcal{V}-\lambda_j) = \dim \operatorname{Ker}(\mathcal{K}(\lambda_1,\lambda_2;\mathcal{H}_0,\mathcal{V})+1).$$

Proof. It suffices to note that the equations

$$(\mathcal{H}_0+\mathcal{V}-\lambda_1)(\mathcal{H}_0+\mathcal{V}-\lambda_2)u = 0,\ u \in D(\mathcal{H}_0),$$

and

$$\mathcal{K}(\lambda_1,\lambda_2;\mathcal{H}_0,\mathcal{V})w + w = 0,\ w \in \mathrm{H},$$

are equivalent for $u = \mathcal{R}(\lambda_1,\lambda_2;\mathcal{H}_0)w$, $w \in \mathrm{H}$. □

Corollary 4.1 *Let $\lambda_1,\lambda_2 \in \mathbf{R}$, $-1<\lambda_1<\lambda_2<1$. Then λ_1 and λ_2 are simultaneously regular points if and only if*

$$\mathrm{vol}\left\{X_\perp \in \mathbf{R}^2 | \dim \operatorname{Ker}(\kappa(\lambda_1,\lambda_2;X_\perp)+1) \geq 1\right\} = 0.$$

5 The ground-levels projection

The unperturbed Hamiltonian can be written as

$$H_0(b) = \begin{pmatrix} I_2 & F(b) \\ F(b) & -I_2 \end{pmatrix}$$

where

$$F(b) := \sum_{j=1}^{3} \sigma_j \Pi_j(b) = \begin{pmatrix} \Pi_3 & a(b) \\ a(b)^* & -\Pi_3 \end{pmatrix},$$

$$a(b) := \Pi_1(b) - i\Pi_2(b),\ a(b)^* := \Pi_1(b) + i\Pi_2(b).$$

The commutation relation $[\Pi_1(b), \Pi_2(b)] = ib$ implies

$$a(b)a(b)^* = \Pi_1(b)^2 + \Pi_2(b)^2 - b,\ a(b)^*a(b) = \Pi_1(b)^2 + \Pi_2(b)^2 + b.$$

Therefore $F(b)^2$ coincides with the Pauli operator

$$F(b)^2 = \left(\sum_{j=1}^{3} \sigma_j \Pi_j(b)\right)^2 = \begin{pmatrix} \Pi(b)^2 - b & 0 \\ 0 & \Pi(b)^2 + b \end{pmatrix} \tag{5.1}$$

where

$$\Pi(b)^2 := \sum_{j=1,2,3} \Pi_j^2.$$

Moreover,

$$H_0(b)^2 = \begin{pmatrix} F(b)^2 + I_2 & 0 \\ 0 & F(b)^2 + I_2 \end{pmatrix}. \tag{5.2}$$

Define the orthogonal projection p_b by

$$(p_b u)(x, y, z) = \int_{\mathbf{R}^2} \mathcal{P}_b(x, y; x', y') u(x', y', z)\, dx'dy',\ u \in L^2(\mathbf{R}^3), \tag{5.3}$$

where

$$\mathcal{P}_b(x, y; x', y') := \frac{b}{2\pi} \exp\left\{-\frac{b}{4}\left[(x - x')^2 + (y - y')^2 + 2i(xy' - yx')\right]\right\}. \tag{5.4}$$

It is essential that $\mathcal{P}_b$ is the integral kernel of the orthogonal projection on $\operatorname{Ker} a(b)^*$ $= \operatorname{Ker} a(b)a(b)^* \subset L^2(\mathbf{R}^2)$. Evidently, p_b commutes with Π_3.
On $L^2(\mathbf{R}^3, \mathbf{C}^4)$ introduce the orthogonal projection

$$P_b := \begin{pmatrix} p_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Obviously, P_b commutes with Π_3 and H_0. Moreover, if $u = (u_1, u_2, u_3, u_4) \in D(H_0)$, we have

$$H_0 P_b u = \begin{pmatrix} 1 & 0 & \Pi_3 & 0 \\ 0 & 0 & 0 & 0 \\ \Pi_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_b u_1 \\ 0 \\ p_b u_3 \\ 0 \end{pmatrix}. \tag{5.6}$$

Put

$$Q_b := \mathrm{Id} - P_b. \tag{5.7}$$

On $\{D(\Pi_3)\}^2 \subset L^2(\mathbf{R}^3;\mathbf{C}^2)$ introduce the operator

$$h_0 := \begin{pmatrix} 1 & \Pi_3 \\ \Pi_3 & -1 \end{pmatrix}. \tag{5.8}$$

Evidently, $\sigma(h_0) = \sigma_{\rm ess}(h_0) = (-\infty,-1]\cup[1,+\infty)$. Note that if we replace Π_3 by $-i\frac{d}{dz}$ in (5.8), we shall obtain the operator χ_0.
Define the operators r and g substituting $H_0(b)$ for h_0 respectively in (4.6) and (4.7). Taking into account (5.6), we find that the spectral theorem for selfadjoint operators entails the following lemma.

Lemma 5.1 *The restrictions of the operators $H_0(b)$ (respectively, R_b and G_b) on $P_bD(H_0(b))$ (respectively, $P_bL^2(\mathbf{R}^3;\mathbf{C}^4)$) are unitarily equivalent to the restrictions of h_0 (respectively, r and g) on $P_bD(h_0)$ (respectively, $p_bL^2(\mathbf{R}^3;\mathbf{C}^2)$).*

6 Preliminary estimates

Lemma 6.1 *Let $\lambda_1,\ \lambda_2 \in \mathbf{R}$, $-1<\lambda_1<\lambda_2<1$. Then the estimates*

$$c_1\||H_0(b)|^{-1}u\|^2 \le \|R_b(\lambda_1,\lambda_2)u\|^2 \le c_2\||H_0(b)|^{-1}u\|^2, \quad \forall u\in L^2(\mathbf{R}^3;\mathbf{C}^4), \tag{6.1}$$

$$c_1\||h_0|^{-1}v\|^2 \le \|r(\lambda_1,\lambda_2)v\|^2 \le c_2\||h_0|^{-1}v\|^2, \quad \forall v\in L^2(\mathbf{R}^3;\mathbf{C}^2), \tag{6.2}$$

$$c_1\||\chi_0|^{-1}w\|^2 \le \|\varrho(\lambda_1,\lambda_2)w\|^2 \le c_2\||\chi_0|^{-1}w\|^2, \quad \forall w\in L^2(\mathbf{R};\mathbf{C}^2), \tag{6.3}$$

hold for some $c_j(\lambda_1,\lambda_2)>0$, $j=1,2$.

Proof. In order to deduce (6.1), it suffices to note that $\|R_b(\lambda_1,\lambda_2)u\|^2 = \|((H_0(b)-\lambda_1)(H_0(b)-\lambda_2))^{-1/2}u\|^2$, and the quantity $|\lambda^2(\lambda-\lambda_1)^{-1}(\lambda-\lambda_2)^{-1}|$ is bounded and strictly positive if $\lambda\in\sigma(H_0(b)) = (-\infty,-1]\cup[1,+\infty)$. Estimates (6.2) and (6.3) are completely analogous. □

Let the matrix $M(X):\mathbf{C}^4\to\mathbf{C}^4$ be defined for $X\in\mathbf{R}^3$. Denote by $|M(X)|$ the norm of $M(X)$, $X\in\mathbf{R}^3$.

Lemma 6.2 *Let $|M|\in L^p(\mathbf{R}^3)$, $p\ge 2$, $\lambda_1,\ \lambda_2\in\mathbf{R}$, $-1<\lambda_1<\lambda_2<1$. Then the estimate*

$$\|MR_bP_b\|_p^p \le bc_3\int_{\mathbf{R}^3}|M(X)|^p\,dX, \quad b>0, \tag{6.4}$$

holds with c_3 which depends on λ_1 and λ_2, but is independent of b and M.

Proof. Evidently,

$$\|MR_bP_b\| \le \|MR_b\| \le c_2\|M|H_0(b)|^{-1}\| \le$$
$$c_2\|\,|H_0(b)|^{-1}\|\|\,|M|\,\|_{L^\infty(\mathbf{R}^3)} = c_2\|\,|M|\,\|_{L^\infty(\mathbf{R}^3)}. \tag{6.5}$$

On the other hand,

$$\|MR_bP_b\|_2^2 = \|Mp_br\|_2^2 \leq c_2^2 \| \, |M| \, p_b |h_0|^{-1}\|_2^2 = 2c_2^2 \| \, |M| \, p_b(\Pi_3^2+1)^{-1/2}\|_2^2. \tag{6.6}$$

Taking into account (5.3)–(5.4) and

$$((\Pi_3^2+1)^{-1/2}u)(x,y,z) = \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{e^{i(z-z')\zeta}}{(\zeta^2+1)^{1/2}} u(x,y,z')\, d\zeta dz',$$

we get

$$\| \, |M| \, p_b(\Pi_3^2+1)^{-1/2}\|_2^2 =$$

$$\frac{b^2}{(2\pi)^3}\int_{\mathbf{R}^3} |M(x,y,z)|^2 dxdydz \int_{\mathbf{R}^2} e^{-\frac{b}{2}\left((x-x')^2+(y-y')^2\right)}\, dx'dy' \int_{\mathbf{R}} \frac{d\zeta}{\zeta^2+1} =$$

$$= \frac{b}{4\pi}\int_{\mathbf{R}^3} |M(X)|^2\, dX. \tag{6.7}$$

Combining (6.6) with (6.7), we obtain

$$\|MR_bP_b\|_2^2 \leq b\frac{c_2^2}{2\pi}\int_{\mathbf{R}^3} |M(X)|^2\, dX. \tag{6.8}$$

Interpolating between (6.5) and (6.8), we find that (6.4) holds with $c_3 = c_2^p/2\pi$. $\square$.

Recall that if $T \in S_p$, $p \geq 1$, then $n_*(\varepsilon; T) \leq \varepsilon^{-p}\|T\|_p^p$, $\varepsilon > 0$.

Corollary 6.1 *Under the assumptions of Lemma* 6.2 *the estimate*

$$n_*(\varepsilon; MR_bP_b) \leq b\, c_3 \varepsilon^{-p}\int_{\mathbf{R}^3} |M(X)|^p\, dX \tag{6.9}$$

holds for every $\varepsilon > 0$ *and* $p \geq 2$.

Lemma 6.3 *Let* $|M| \in L^3(\mathbf{R}^3)$.
(i) *There exists a constant* c_4 *such that for every* $\varepsilon > 0$ *we have*

$$n_*(\varepsilon; MR_bQ_b) \leq c_4\varepsilon^{-3}\int_{\mathbf{R}^3} |M(X)|^3\, dX. \tag{6.10}$$

(ii) *Moreover, for every* $\varepsilon > 0$, λ_1, $\lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$, *and* $|M| \in L^3(\mathbf{R}^3)$, *there exists a number* b_0 *such that* $b \geq b_0$ *entails*

$$n_*(\varepsilon; MR_bQ_b) = 0. \tag{6.11}$$

Proof. By Lemma 6.1 we have

$$n_*(\varepsilon; MR_bQ_b) \le n_*(\varepsilon c_2^{-1}; |M||H_0(b)|^{-1}Q_b), \quad \varepsilon > 0. \tag{6.12}$$

Further, (5.1)-(5.2) entail

$$|H_0(b)|^{-1}Q_b \le (\Pi^2 + b)^{-1/2}Q_b \le (\Pi^2 + b)^{-1/2}I_4.$$

Moreover, the operators $|H_0(b)|^{-1}$, Q_b and $(\Pi^2 + b)^{-1/2}I_4$ are pairwise commuting. Therefore, we have

$$n_*(\varepsilon c_2^{-1}; |M||H_0(b)|^{-1}Q_b) \le 4n_*(\varepsilon c_2^{-1}; |M|(\Pi^2 + b)^{-1/2}). \tag{6.13}$$

The classical Birman-Schwinger principle (see (4.1)) entails

$$n_*(\varepsilon c_2^{-1}; |M|(\Pi^2 + b)^{-1/2}) = N(-b; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2) \le N(0; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2), \tag{6.14}$$

while the magnetic version of the Cwickel-Lieb-Rozenblioum estimate implies

$$N(0; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2) \le c_5c_2^3\varepsilon^{-3} \int_{\mathbf{R}^3} |M(X)|^3\, dX \tag{6.15}$$

where c_5 is independent of M and ε (see [1, Theorem 2.15]).
Now, the combination of (6.12)–(6.15) immediately yields (6.10) with $c_4 = c_5c_2^3$.
On the other hand, by the Kato–Simon inequality we have

$$\|\, |M|\, (\Pi^2 + b)^{-1/2}\| \le \|\, |M|\, (-\Delta + b)^{-1/2}\|. \tag{6.16}$$

Since $|M| \in L^3(\mathbf{R}^3)$, the multiplier by $|M|^2$ is -Δ-form-compact. Therefore

$$\lim_{b\to\infty} \|\, |M|\, (-\Delta + b)^{-1/2}\| = 0. \tag{6.17}$$

Fix $\varepsilon > 0$, and taking into account (6.16)–(6.17), choose b_0 so that $b \ge b_0$ entails

$$\|\, |M|\, (\Pi^2 + b)^{-1/2}\| < \varepsilon c_2^{-1}. \tag{6.18}$$

Now, (6.12) and (6.13) combined with (6.18) imply (6.11). □

Corollary 6.2 *Let $M \in L^3(\mathbf{R}^3)$. Then for every $\varepsilon > 0$ and $b > 0$ we have*

$$n_*(\varepsilon; MR_b) \le (c_3b + c_4)\left(\frac{\varepsilon}{2}\right)^{-3} \int_{\mathbf{R}^3} |M(X)|^3\, dX. \tag{6.19}$$

Proof. Since $Q_b + P_b = \mathrm{Id}$, we have

$$n_*(\varepsilon; MR_b) = n_*(\varepsilon; MR_bP_b + MR_bQ_b) \le n_*(\varepsilon/2; MR_bP_b) + n_*(\varepsilon/2; MR_bQ_b).$$

Applying (6.9) with $p = 3$ and (6.10), we get (6.19). □

Remark. In most cases we shall apply Lemmas 6.2 - 6.3 and Corollaries 6.1 - 6.2 with $M = V\, I_4$.

7 Proof of Propositions 2.1 – 2.2

Let $V : \mathbf{R}^3 \to \mathbf{R}$ be a measurable function. Fix $\varepsilon > 0$ and set

$$V_1(X) = V_{1,\varepsilon}(X) = \begin{cases} V(X) \text{ if } |V(X)| > \varepsilon, \\ 0 \text{ otherwise,} \end{cases} \tag{7.1}$$

$$V_2(X) = V_{2,\varepsilon}(X) = V(X) - V_{1,\varepsilon}(X). \tag{7.2}$$

It is easy to check that $V \in \mathcal{L}$ is equivalent to $V_{1,\varepsilon} \in L^3(\mathbf{R}^3)$ for all $\varepsilon > 0$. In what follows if $V \in \mathcal{L}$ and $\varepsilon > 0$ we shall choose the decomposition $V = V_1 + V_2$ with $V_1 \in L^3(\mathbf{R}^3)$ and $\sup_{X\in\mathbf{R}^3} |V_2(X)| \leq \varepsilon$, as in (7.1)–(7.2), and shall call it briefly the *ε-decomposition* of V.

Lemma 7.1 *Let $V \in \mathcal{L}$. Then for almost every $X_\perp \in \mathbf{R}^2$ the operator $V(X_\perp, .)\chi_0^{-1}$ is compact in $L^2(\mathbf{R})$.*

Proof. Fix ε and write the ε-decomposition of V. Choose $X_\perp \in \mathbf{R}^2$ so that

$$\int_{\mathbf{R}} |V_1(X_\perp, z)|^3 \, dz < \infty. \tag{7.3}$$

Evidently, the complement of the set of $X_\perp$ satisfying (7.3), is a null-set. Moreover, (7.3) implies that the operator $V_1(X_\perp, .)\chi_0^{-1}$ is compact.
Now, pick a sequence ε_n such that $\varepsilon_n > 0$, and $\lim_{n\to\infty} \varepsilon_n = 0$. Write the ε_n-decomposition of $V = V_1^{(n)} + V_2^{(n)}$ with $V_j^{(n)} := V_{j,\varepsilon_n}$, $j = 1, 2$. Fix $X_\perp \in \mathbf{R}^2$ such that $\int_{\mathbf{R}} |V_1^{(n)}(X_\perp, z)|^3 \, dz < \infty$ for all n. The complement of such $X_\perp$ is again a null-set being a countable union of null-sets. The operators $V_1^{(n)}(X_\perp, .)\chi_0^{-1}$ are compact, and we have

$$\|V_1(X_\perp, .)\chi_0^{-1} - V_1^{(n)}(X_\perp, .)\chi_0^{-1}\| = \|V_2^{(n)}(X_\perp, .)\chi_0^{-1}\| \leq \varepsilon_n.$$

Since the operator $V(X_\perp, .)\chi_0^{-1}$ can be approximated in norm by compact operators, it is a compact operator itself. □

Remark. Lemma 7.1 entails immediately Proposition 2.1.

Lemma 7.2 *Let $v \in L^p(\mathbf{R})$, $p \geq 2$. Then we have*

$$\|v\varrho\|_p^p \leq c_6 \int_{\mathbf{R}} |v(z)|^p \, dz \tag{7.4}$$

where $c_6 = c_6(p)$ is independent of v.

Proof. Applying Lemma 6.1, we get

$$\|v\varrho\|_p^p \leq c_2^p \|v|\chi_0|^{-1}\|_p^p. \tag{7.5}$$

Evidently,

$$\|v|\chi_0|^{-1}\|_p^p = 4\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\|_p^p. \tag{7.6}$$

If $v \in L^\infty(\mathbf{R})$, we have

$$\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\| \leq \sup_{\zeta\in\mathbf{R}}(\zeta^2+1)^{-1/2}\,\|v\|_{L^\infty(\mathbf{R})} = \|v\|_{L^\infty(\mathbf{R})}. \tag{7.7}$$

If $v \in L^2(\mathbf{R})$, we have

$$\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\|_2^2 = \frac{1}{2\pi}\int_{\mathbf{R}}\frac{d\zeta}{\zeta^2+1}\,\|v\|_{L^2(\mathbf{R})}^2 = \frac{1}{2}\|v\|_{L^2(\mathbf{R})}^2. \tag{7.8}$$

Interpolating between (7.7) and (7.8), and bearing in mind (7.5) and (7.6), we find that (7.4) holds with $c_6 = 2c_2^p$. □

Corollary 7.1 *Let $V \in L^3(\mathbf{R}^3)$. Then the estimate*

$$\int_{\mathbf{R}^2} n_*(\varepsilon; V(X_\perp,.)\varrho)\,dX_\perp \leq c_6\varepsilon^{-3}\int_{\mathbf{R}^3}|V(X)|^3\,dX \tag{7.9}$$

holds for each $\varepsilon > 0$ with $c_6 = c_6(3)$.

Proof. Fix $X_\perp \in \mathbf{R}^2$ for which $\int_{\mathbf{R}^2}|V(X_\perp,z)|^3\,dz < \infty$; the complement of the set of such $X_\perp$ is a null set. Applying (7.4), we get

$$n_*(\varepsilon; V(X_\perp,.)\varrho) \leq \varepsilon^{-3}\|V(X_\perp,.)\varrho\|_3^3 \leq c_6\varepsilon^{-3}\int_{\mathbf{R}}|V(X_\perp,z)|^3\,dz.$$

Integrating with respect to $X_\perp \in \mathbf{R}^2$, we get (7.9). □

Corollary 7.2 *Let $V \in L^3(\mathbf{R}^3)$, $-1 < \lambda_1 < \lambda_2 < 1$. Then we have*

$$\mathcal{D}(\lambda_1,\lambda_2) \leq c_7\varepsilon^{-3}\int_{\mathbf{R}^3}|V(X)|^3\,dX \tag{7.10}$$

with $c_7 = c_7(\lambda_1,\lambda_2) = 2^4(c_2(\lambda_1,\lambda_2)\|\gamma(\lambda_1,\lambda_2)\|)^3$.

Proof. First, by (4.10) and $\varrho V(X_\perp,.)^2\varrho \geq 0$, we have

$$\mathcal{D}(\lambda_1,\lambda_2) \leq \int_{\mathbf{R}^2} n_-(1; 2\mathrm{Re}\gamma V(X_\perp,.)\varrho)dX_\perp \leq \int_{\mathbf{R}^2} n_*(1; 2\|\gamma\|V(X_\perp.,)\varrho)dX_\perp.$$

Further, Corollary 7.1 implies

$$\int_{\mathbf{R}^2} n_*(1; 2\|\gamma\|V(X_\perp,.)\varrho)dX_\perp \leq c_6 2^3\|\gamma\|^3\int_{\mathbf{R}^3}|V(X)|^3\,dX. \tag{7.11}$$

Inserting the value of c_6 into (7.11), we obtain (7.10). $\square$

Remark. Proposition 2.2 is implied almost immediately by Corollary 7.2. In order to see that, we assume that $-1 < \lambda_1 < \lambda_2 < 1$, fix $\varepsilon > 0$ such that $\lambda_1 - \varepsilon > -1$ and $\lambda_2 + \varepsilon < 1$, and write the ε-decomposition of V. Then we have

$$\mathcal{D}_V(\lambda_1, \lambda_2) \le \mathcal{D}_{V_1}(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon),$$

and by (7.10)

$$\mathcal{D}_{V_1}(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \le c_7(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \int_{\mathbf{R}^3} |V_1(X)|^3 \, dX.$$

Therefore $\mathcal{D}_V(\lambda_1, \lambda_2) < \infty$.

8 Proof of Theorem 2.1

Let $-1 < \lambda_1 < \lambda_2 < 1$. Introduce the operator

$$k_b \equiv k_b(\lambda_1, \lambda_2) := r(\lambda_1, \lambda_2) p_b V^2 p_b r(\lambda_1, \lambda_2) + 2\mathrm{Re}\; g(\lambda_1, \lambda_2) p_b V p_b r(\lambda_1, \lambda_2)$$

where the operator p_b is defined by (5.3), while the operators $r(\lambda_1, \lambda_2)$ and $g(\lambda_1, \lambda_2)$ are introduced at the end of Section 5.
It is easy to check that if $V \in L^p(\mathbf{R}^3)$ then $V p_b r \in S_p$, $p \ge 2$; hence, $V p_b r$ itself as well as $r p_b V^2 p_b r$, $g p_b V p_b r$ and $r p_b V p_b g$ are Hilbert-Schmidt operators.

Proposition 8.1 *Let $V \in C_0^\infty(\mathbf{R}^3)$, $-1 < \lambda_1 < \lambda_2 < 1$. Then the asymptotic relations*

$$\lim_{b \to \infty} b^{-1} \mathrm{Tr}\; k_b(\lambda_1, \lambda_2)^l = \frac{1}{2\pi} \int_{\mathbf{R}^2} \kappa(\lambda_1, \lambda_2; X_\perp)^l \, dX_\perp$$

are valid for every integer $l \ge 2$.

Proof. Throughout the proof the parameters λ_1 and λ_2 are fixed, and we omit them in the notations.
For $l \ge 1$ write

$$k_b^l = \sum_{j=1}^{3^l} k_{j,l}(b), \; \kappa(X_\perp)^l = \sum_{j=1}^{3^l} \kappa_{j,l}(X_\perp),$$

where the terms $k_{j,l}$ and $\kappa_{j,l}$, $j = 1, \dots, 3^l$, are defined recurrently:

$$k_{1,1}(b) := r p_b V^2 p_b r, \; k_{2,1}(b) := g p_b V p_b r, \; k_{3,1}(b) := r p_b V p_b g,$$

$$\kappa_{1,1}(X_\perp) := \varrho V(X_\perp, .)^2 \varrho, \; \kappa_{2,1}(X_\perp) := \gamma V(X_\perp, .) \varrho, \; \kappa_{3,1}(X_\perp) := \varrho V(X_\perp, .) \gamma,$$

$$k_{j,l}(b) = \begin{cases} k_{1,1}(b)\ k_{j,l-1}(b),\ j = 1,\ldots,3^{l-1}, \\ k_{2,1}(b)\ k_{j,l-1}(b),\ j = 3^{l-1}+1,\ldots,2.3^{l-1}, \quad l \geq 2, \\ k_{3,1}(b)\ k_{j,l-1}(b),\ j = 2.3^{l-1}+1,\ldots,3^l, \end{cases}$$

$$\kappa_{j,l}(X_\perp) = \begin{cases} \kappa_{1,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp),\ j = 1,\ldots,3^{l-1}, \\ \kappa_{2,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp),\ j = 3^{l-1}+1,\ldots,2.3^{l-1}, \quad l \geq 2. \\ \kappa_{3,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp),\ j = 2.3^{l-1}+1,\ldots,3^l, \end{cases}$$

The operators $k_{j,l}(b)$, $j = 1,\ldots,l$, $l \geq 2$, can be written in the form

$$k_{j,l} = E_{j,l}^- W_{1,j,l} T_{1,j,l} \ldots W_{l-1,j,l} T_{l-1,j,l} W_{l,j,l} E_{j,l}^+$$

where the operators $E_{j,l}^-$ and $E_{j,l}^+$ coincide either with r or with g, the operators $W_{s,j,l}$, $s = 1,\ldots,l$, coincide either with $p_b V^2 p_b$ or with $p_b V p_b$, and the operators $T_{s,j,l}$, $s = 1,\ldots,l-1$, coincide either with r^2, or with gr, or with g^2. Note that among the operators $T_{s,j,l}$, $s = 1,\ldots,l-1$, and $E_{j,l}^+ \times E_{j,l}^-$, there are either at least one operator r^2, or at least two operators gr.
Analogously,

$$\kappa_{j,l}(X_\perp) = \epsilon_{j,l}^- \omega_{1,j,l}(X_\perp) \tau_{1,j,l} \ldots \omega_{l-1,j,l}(X_\perp) \tau_{l-1,j,l} \omega_{l,j,l}(X_\perp) \epsilon_{j,l}^+$$

where $\epsilon_{j,l}^\pm = \varrho$ if $E_{j,l}^\pm = r$ and $\epsilon_{j,l}^\pm = \gamma$ if $E_{j,l}^\pm = g$, $\omega_{s,j,l}(X_\perp) = V(X_\perp,.)^2$ if $W_{s,j,l} = p_b V^2 p_b$, and $\omega_{s,j,l}(X_\perp) = V(X_\perp,.)$ if $W_{s,j,l} = p_b V p_b$, $s = 1,\ldots,l$, $\tau_{s,j,l} = \varrho^2$ if $T_{s,j,l} = r^2$, $\tau_{s,j,l} = \varrho\gamma$ if $T_{s,j,l} = rg$, and $\tau_{s,j,l} = \gamma^2$ if $T_{s,j,l} = g^2$, $s = 1,\ldots,l-1$.
Obviously,

$$\mathrm{Tr}\ k_b^l = \sum_{j=1}^{3^l} \mathrm{Tr}\ k_{j,l}(b),$$

$$\int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa(X_\perp)^l dX_\perp = \sum_{j=1}^{3^l} \int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\ dX_\perp,\ l \geq 2.$$

Hence, it suffices to prove that

$$\lim_{b\to\infty} b^{-1} \mathrm{Tr}\ k_{j,l}(b) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\ dX_\perp,\ j = 1,\ldots,3^l,\ l \geq 2. \tag{8.1}$$

It is not difficult to show that

$$\mathrm{Tr}\ k_{j,l} = \int_{\mathbf{R}^{3l}} \Pi'^l_{s=1} w_{s,j,l}(x_{s+1}, y_{s+1}, \zeta_{s+1} - \zeta_s) \mathcal{P}_b(x_{s+1}, y_{s+1}; x_s, y_s) \times$$
$$\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\ \Pi^l_{s=1} dx_s dy_s d\zeta_s \tag{8.2}$$

where

$$w_{s,j,l}(x,y,\zeta) = \begin{cases} \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iz\zeta} V^2(x,y,z) dz \quad \text{if} \quad W_{s,j,l} = p_b V^2 p_b, \\ \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iz\zeta} V(x,y,z) dz \quad \text{if} \quad W_{s,j,l} = p_b V p_b, \end{cases}$$

$\mathcal{P}_b$ is introduced in (5.4), $t_{s,j,l}(\zeta)$, $s=1,\dots,l-1$, coincides with the matrix-valued symbol of the operator $T_{s,j,l}$, and $t_{l,j,l}(\zeta)$ is the matrix-valued symbol of the operator $E^+_{j,l}\times E^-_{j,l}$. Moreover, the notation $\Pi'^l_{s=1}$ means that in the product of l factors the variables x_{l+1}, y_{l+1}, and ζ_{l+1}, should be set equal respectively to x_1, y_1, and ζ_1.
Analogously, we have

$$\mathrm{Tr}\ \kappa_{j,l}(X_\perp)\equiv \mathrm{Tr}\ \kappa_{j,l}(x,y)=$$

$$\int_{\mathbf{R}^l}\Pi'^l_{s=1}w_{s,j,l}(x,y,\zeta_{s+1}-\zeta_s)\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}d\zeta_s,\ X_\perp\equiv(x,y)\in\mathbf{R}^2. \tag{8.3}$$

In order to prove (8.1), we insert (5.4) into (8.2), and obtain

$$\mathrm{Tr}\ k_{j,l}=\frac{b^l}{(2\pi)^l}\int_{\mathbf{R}^{3l}}\Pi'^l_{s=1}w_{s,j,l}(x_{s+1},y_{s+1},\zeta_{s+1}-\zeta_s)\times$$

$$\exp\left\{-\frac{b}{4}\left[(x_{s+1}-x_s)^2+(y_{s+1}-y_s)^2+2i(x_{s+1}y_s-y_{s+1}x_s)\right]\right\}$$

$$\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}dx_sdy_sd\zeta_s.$$

Change the variables

$$x_1=x'_1,\ y_1=y'_1,$$

$$x_s=b^{-1/2}x'_s+x'_1,\ y_s=b^{-1/2}y'_s+y'_1,\ s=2,\dots,l.$$

Note that the corresponding Jacobian is equal to b^{1-l}. Thus we get

$$\mathrm{Tr}\ k_{j,l}=\frac{b}{(2\pi)^l}\int_{\mathbf{R}^{3l}}w_{l,j,l}(x'_1,y'_1,\zeta_1-\zeta_l)e^{\Phi(x'_2,\dots,x'_l,y'_2,\dots,y'_l)}$$

$$\Pi^{l-1}_{s=1}w_{s,j,l}(x'_1+b^{-1/2}x'_{s+1},y'_1+b^{-1/2}y'_{s+1},\zeta_{s+1}-\zeta_s)\ \mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\ \Pi^l_{s=1}dx'_sdy'_sd\zeta_s,$$

where

$$\Phi(x_2,\dots,x_l,y_2,\dots,y_l):=$$

$$-\frac{1}{4}\left\{{x_2}^2+{y_2}^2+{x_l}^2+{y_l}^2+\sum_{s=2}^{l-1}\Big((x_{s+1}-x_s)^2+(y_{s+1}-y_s)^2+2i(x_{s+1}y_s-y_{s+1}x_s)\Big)\right\}.$$

Therefore,

$$\lim_{b\to\infty}b^{-1}\mathrm{Tr}\ k_{j,l}(b)=$$

$$\frac{1}{(2\pi)^l}\int_{\mathbf{R}^{3l}}e^{\Phi(x'_2,\dots,x'_l,y'_2,\dots,y'_l)}\Pi'^l_{s=1}w_{s,j,l}(x'_1,y'_1,\zeta_{s+1}-\zeta_s)\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}dx'_sdy'_sd\zeta_s. \tag{8.4}$$

Changing the variables in the integral at the right hand side of (8.4)

$$x'_1=x_1,\ y'_1=y_1,$$

$$x'_s=x_s-x_1,\ y'_s=y_s-y_1,\ s=2,\dots,l,$$

we get

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ k_{j,l}(b) = \int_{\mathbf{R}^{3l}} \Pi'^{l}_{s=1} w_{s,j,l}(x_1, y_1, \zeta_{s+1} - \zeta_s)$$

$$\mathcal{P}_1(x_{s+1}, y_{s+1}; x_s, y_s)\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\Pi^l_{s=1} dx_s dy_s d\zeta_s. \tag{8.5}$$

Since

$$\int_{\mathbf{R}^2} \mathcal{P}_1(x, y; x'', y'')\mathcal{P}_1(x'', y''; x', y')dx''dy'' = \mathcal{P}_1(x, y; x', y'),\ x, y, x', y' \in \mathbf{R},$$

$$\mathcal{P}_1(x, y; x, y) = (2\pi)^{-1},\ x, y \in \mathbf{R},$$

we find that (8.5) is equivalent to

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ k_{j,l}(b) =$$

$$\frac{1}{2\pi}\int_{\mathbf{R}^{l+2}} \Pi'^{l-1}_{s=1} w_{s,j,l}(x_1, y_1, \zeta_{s+1} - \zeta_s)\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\, \Pi^l_{s=1} d\zeta_s\, dx_1 dy_1 =$$

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\, dX_\perp,\ j = 1, \dots, l,\ l \geq 2,$$

(see (8.3)), which is identical to (8.1). □

Set

$$\nu(s) := \begin{cases} \frac{1}{2\pi}\int_{\mathbf{R}^2} n_-(-s; \kappa(X_\perp))\, dX_\perp,\ s < 0, \\ -\frac{1}{2\pi}\int_{\mathbf{R}^2} n_+(s; \kappa(X_\perp))\, dX_\perp,\ s > 0, \end{cases} \tag{8.6}$$

Note that $s \neq 0$ is a continuity point of ν if and only if

$$\mathrm{vol}\ \left\{X_\perp \in \mathbf{R}^2 | \dim\ \mathrm{Ker}\ (\kappa(X_\perp) - s) \geq 1\right\} = 0.$$

Corollary 8.1 *Let $t < 0$ be a continuity point of ν. Then we have*

$$\lim_{b\to\infty} b^{-1} n_-(-t; k_b) = \nu(t).$$

The corollary follows immediately from Proposition 8.1 and Lemma 3.1 with $T(b) = k_b$, ν defined as in (8.6), and $t_0 = \|r\|^2\|V\|^2_{L^\infty(\mathbf{R})} + 2\|g\|\|r\|\|V\|_{L^\infty(\mathbf{R})}$.

Proposition 8.2 *Let $V \in C_0^\infty(\mathbf{R}^3)$. Assume that $t < 0$ is a continuity point of ν. Then we have*

$$\lim_{b\to\infty} b^{-1} n_-(-t; K_b) = \nu(t). \tag{8.7}$$

Proof. Evidently

$$n_-(-t; K_b) \geq n_-(-t; P_b K_b P_b) = n_-(-t; k_b).$$

Applying Corollary 8.1, we get

$$\liminf_{b\to\infty} b^{-1}n_-(-t;K_b) \geq \liminf_{b\to\infty} b^{-1}n_-(-t;k_b) = \lim_{b\to\infty} b^{-1}n_-(-t;k_b) = \nu(t). \qquad (8.8)$$

On the other hand we have

$$K_b = P_bK_bP_b + Q_bK_bQ_b + 2\text{Re } P_bK_bQ_b =$$

$$P_bK_bP_b + Q_bK_bQ_b + 2\text{Re } P_bR_bV^2R_bQ_b + 2\text{Re } P_bG_bVR_bQ_b + 2\text{Re } P_bR_bVG_bQ_b.$$

Note that

$$R_bVG_b = G_bVR_b + R_bJR_b$$

where

$$J := [V, H_0(b) - \frac{1}{2}(\lambda_1+\lambda_2)] = [V, H_0(b)] = i\left(\frac{\partial V}{\partial x}\alpha_1 + \frac{\partial V}{\partial y}\alpha_2 + \frac{\partial V}{\partial z}\alpha_3\right).$$

It is essential that J is independent of b.
Apply the estimates

$$K_b = P_bK_bP_b + Q_bK_bQ_b + 2\text{Re } P_bR_bV^2R_bQ_b + 4\text{Re } P_bG_bVR_bQ_b + 2\text{Re } P_bR_bJR_bQ_b \geq$$

$$P_bK_bP_b + Q_bK_bQ_b - \varepsilon P_bR_bV^2R_bP_b - \varepsilon^{-1}Q_bR_bV^2R_bQ_b -$$

$$2\varepsilon P_bG_b^2P_b - 2\varepsilon^{-1}Q_bR_bV^2R_bQ_b - \varepsilon P_bR_b^2P_b - \varepsilon^{-1}Q_bR_bJ^*JR_bQ_b \geq$$

$$P_b(K_b - \varepsilon(R_bV^2R_b + 2G_b^2 + R_b^2))P_b - Q_b(\varepsilon G_b^2 + \varepsilon^{-1}R_b(4V^2 + J^*J)R_b)Q_b, \; \varepsilon > 0. \quad (8.9)$$

Now fix $\mu \in (0,-t)$, and choose ε so small that we have $\varepsilon(3\|G_b\|^2 + \|R_b\|^2) \leq \mu/3$; hence $\varepsilon\|2P_bG_b^2P_b + P_bR_b^2P_b + Q_bG_b^2Q_b\| \leq \mu/3$. Then (8.9) entails

$$n_-(-t;K_b) \leq n_-(-t-\mu; P_bK_bP_b) +$$

$$n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b) + n_+(\mu\varepsilon/3; Q_bR_b(4V^2 + J^*J)R_bQ_b). \qquad (8.10)$$

Lemma 6.3(ii) combined with the estimate

$$n_+(2\delta^2; Q_bR_b(4V^2 + J^*J)R_bQ_b) \leq n_*(\delta/2; VR_bQ_b) + n_*(\delta; JR_bQ_b), \quad \delta > 0,$$

implies that the quantity $n_+(\mu\varepsilon/3; Q_bR_b(4V^2+J^*J)R_bQ_b)$ vanishes for b large enough. Hence, (8.10) entails

$$\limsup_{b\to\infty} b^{-1}n_-(-t;K_b) \leq \limsup_{b\to\infty} b^{-1}n_-(-t-\mu; P_bK_bP_b) +$$

$$\limsup_{b\to\infty} b^{-1}n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b), \; \mu \in (0,-t), \; \varepsilon > 0. \qquad (8.11)$$

Corollary 6.1 combined with the estimate $n_+(\delta^2; P_bR_bV^2R_bP_b) \le n_*(\delta; VR_bP_b)$, $\delta > 0$, implies

$$\limsup_{b\to\infty} b^{-1}n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b) \le c_3\left(\frac{3\varepsilon}{\mu}\right)^{3/2}\int_{\mathbf{R}^3}|V|^3\,dX.$$

Letting $\varepsilon \downarrow 0$, we find that (8.11) entails

$$\limsup_{b\to\infty} b^{-1}n_-(-t; K_b) \le \limsup_{b\to\infty} b^{-1}n_-(-t-\mu; P_bK_bP_b),\ \mu\in(0,-t). \qquad (8.12)$$

Now choose a sequence $\{\mu_l\}_{l\ge1}$ such that $\mu_l \in (0,-t)$, $\lim_{l\to\infty}\mu_l = 0$, and all the points $-t-\mu_l$ are continuity points of ν. Then Corollary 8.1 implies

$$\limsup_{b\to\infty} b^{-1}n_-(-t-\mu_l; P_bK_bP_b) = \lim_{b\to\infty} b^{-1}n_-(-t-\mu_l; k_b) = \nu(t+\mu_l). \qquad (8.13)$$

Letting $l\to\infty$, we find that (8.11)–(8.13) entail

$$\limsup_{b\to\infty} b^{-1}n_-(-t; K_b) \le \nu(t). \qquad (8.14)$$

The combination of (8.8) and (8.14) immediately yields (8.7). □

Proposition 8.3 *Let $V \in L^3(\mathbf{R}^3)$. Assume that $t<0$ is a continuity point of ν. Then* (8.7) *remains valid.*

Proof. Pick a sequence $\{\delta_l\}_{l\ge1}$, $\lim_{l\to\infty}\delta_l = 0$, and write $V = V_0 + V_1$, where $V_0 = V_{0,l} \in C_0^\infty(\mathbf{R}^3)$, $V_1 = V_{1,l} \in L^3(\mathbf{R}^3)$, and $\|V_{1,l}\|_{L^3(\mathbf{R}^3)} \le \delta_l$.
Introduce the operators $K_{b,0}$, $k_{b,0}$ and $\kappa_0(X_\perp)$, replacing V by $V_{0,l}$. Analogously, define the function $\nu_l \equiv \nu_{0,l}$ substituting κ for κ_0. Choose the sequence $\{\varepsilon_r\}$ such that $0<\varepsilon_r<\min\{1,-t/2\}$, $\lim_{r\to\infty}\varepsilon_r = 0$, and the points $-t\pm\varepsilon_r$ are continuity points of all functions $\nu_{0,l}$. Evidently,

$$K_b \ge (1-\varepsilon_r^2)K_{b,0} + (1-\varepsilon_r^{-2})R_bV_1^2R_b + 2\mathrm{Re}G_bV_1R_b + 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b,$$

$$K_b \le (1+\varepsilon_r^2)K_{b,0} + (1+\varepsilon_r^{-2})R_bV_1^2R_b + 2\mathrm{Re}G_bV_1R_b - 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b,$$

and, hence,

$$n_-(-t;K_b) \le n_-(-t-\varepsilon_r; K_{b,0}) + n_-(\varepsilon_r/3; (1-\varepsilon_r^{-2})R_bV_1^2R_b) +$$
$$n_-(\varepsilon_r/3; 2\mathrm{Re}\ G_bV_0R_b) + n_-(\varepsilon_r/3; 2\varepsilon_r^2\mathrm{Re}\ G_bV_1R_b),$$
$$n_-(-t;K_b) \ge n_-(-t+\varepsilon_r; K_{b,0}) - n_+(\varepsilon_r/3; (1+\varepsilon_r^{-2})R_bV_1^2R_b) -$$
$$n_+(\varepsilon_r/3; 2\mathrm{Re}G_bV_1R_b) - n_-(\varepsilon_r/3; 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b).$$

Utilizing Corollary 6.2 and Proposition 8.2, we get

$$\limsup_{b\to\infty} b^{-1}n_+(-t;K_b) \le \nu_{0,l}(t+\varepsilon_r) + c_8\delta_l^3 + c_9\varepsilon_r^3\int_{\mathbf{R}^3}|V_0|^3\,dX, \qquad (8.15)$$

$$\liminf_{b\to\infty} b^{-1} n_+(-t; K_b) \geq \nu_{0,l}(t-\varepsilon_r) - c_8\delta_l^3 - c_9\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3 \, dX, \tag{8.16}$$

where c_8 depends on ε_r but is independent on δ_l, while c_9 is independent of both ε_r and δ_l.
Similarly, using Corollaries 7.1 and 7.2, we obtain the estimates

$$\nu_{0,l}(t+\varepsilon_r) \leq \nu(t+2\varepsilon_r) + c_8'\delta_l^3 + c_9'\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3 \, dX, \tag{8.17}$$

$$\nu_{0,l}(t-\varepsilon_r) \geq \nu(t-2\varepsilon_r) - c_8'\delta_l^3 - c_9'\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3 \, dX, \tag{8.18}$$

where c_8' depends on ε_r but is independent on δ_l, while c_9' is independent of both ε_r and δ_l.
Letting at first $l \to \infty$ (hence, $\delta_l \downarrow 0$), and then $r \to \infty$ (hence, $\varepsilon_r \downarrow 0$), in (8.15) - (8.18), and taking into account that t is a continuity point of ν, we obtain (8.7). $\Box$

Using Lemma 4.1, Corollary 4.1, and Proposition 8.3 with $t = -1$, we deduce the following corollary.

Corollary 8.2 *Let the hypotheses of Theorem* 2.1 *hold. Assume in addition* $V \in L^3(\mathbf{R}^3)$. *Then* (2.1) *holds.*

In order to complete the proof of Theorem 2.1 it remains to show that we can approximate $V \in \mathcal{L}$ by $V \in L^3(\mathbf{R}^3)$.
Let $\lambda_1, \lambda_2 \in \mathbf{R}^3$, $-1 < \lambda_1 < \lambda_2 < 1$, be regular points. Fix $\varepsilon > 0$ such that $\lambda_1 - 3\varepsilon > -1$, $\lambda_2 + 3\varepsilon < 1$, $\lambda_1 + 3\varepsilon < \lambda_2 - 3\varepsilon$, and write the ε-decomposition of V. Evidently,

$$\mathcal{N}(\lambda_1+\varepsilon, \lambda_2-\varepsilon; H_0(b)+V_1) \leq \mathcal{N}(\lambda_1, \lambda_2; H(b)) \leq \mathcal{N}(\lambda_1-\varepsilon, \lambda_2+\varepsilon; H_0(b)+V_1). \tag{8.19}$$

Now chose the numbers $\varepsilon_j \in [0, \varepsilon)$ such that

$$\mathrm{vol}\left\{X_\perp \in \mathbf{R}^2 | \dim \mathrm{Ker}\ (\chi_0 + V_1(X_\perp, .) - (\lambda_j \pm \varepsilon \pm \varepsilon_j)) \geq 1\right\} = 0, \quad j = 1, 2.$$

Applying Corollary 8.2, we deduce from (8.19) the following estimates

$$\begin{aligned}
\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) &\leq \limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1-\varepsilon, \lambda_2+\varepsilon; H_0(b)+V_1) \leq \\
&\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1-\varepsilon-\varepsilon_1, \lambda_2+\varepsilon+\varepsilon_2; H_0(b)+V_1) = \\
&\mathcal{D}_{V_1}(\lambda_1-\varepsilon-\varepsilon_1, \lambda_2+\varepsilon+\varepsilon_2) \leq \mathcal{D}_{V_1}(\lambda_1-2\varepsilon, \lambda_2+2\varepsilon), \qquad (8.20)\\
\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) &\geq \liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1+\varepsilon, \lambda_2-\varepsilon; H_0(b)+V_1) \geq \\
&\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1+\varepsilon+\varepsilon_1, \lambda_2-\varepsilon-\varepsilon_2; H_0(b)+V_1) =
\end{aligned}$$

$$\mathcal{D}_{V_1}(\lambda_1+\varepsilon+\varepsilon_1, \lambda_2-\varepsilon-\varepsilon_2) \leq \mathcal{D}_{V_1}(\lambda_1+2\varepsilon, \lambda_2-2\varepsilon). \tag{8.21}$$

Finally, note the obvious inequalities

$$\mathcal{D}_{V_1}(\lambda_1-2\varepsilon, \lambda_2+2\varepsilon) \leq \mathcal{D}_V(\lambda_1-3\varepsilon, \lambda_2+3\varepsilon),$$

$$\mathcal{D}_{V_1}(\lambda_1+2\varepsilon, \lambda_2-2\varepsilon) \geq \mathcal{D}_V(\lambda_1+3\varepsilon, \lambda_2-3\varepsilon). \tag{8.22}$$

Putting together (8.19)–(8.22), we get

$$\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) \leq \mathcal{D}_V(\lambda_1-3\varepsilon, \lambda_2+3\varepsilon), \tag{8.23}$$

$$\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) \geq \mathcal{D}_V(\lambda_1+3\varepsilon, \lambda_2-3\varepsilon). \tag{8.24}$$

Letting $\varepsilon \downarrow 0$ in (8.23)–(8.24), and bearing in mind that λ_1 and λ_2 are regular points, we come to (2.1).

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References

[1] J.Avron, I.Herbst, B.Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke. Math. J. **45** (1978), 847-883.

[2] M.Š.Birman, *On the spectrum of singular boundary value problems*, Mat. Sbornik **55** (1961) 125-174 (Russian); Engl. transl. in Amer. Math. Soc. Transl., (2) **53** (1966), 23-80.

[3] M.Sh.Birman, A.Laptev, *Discrete spectrum of the perturbed Dirac operator*, Ark.Mat. **32** (1994), 13-32.

[4] C.Cancelier, P.Lévy-Bruhl, J.Nourrigat, *Remarks on the spectrum of the Dirac operator*, Acta Appl.Math. **45** (1996), 349-364.

[5] W.D.Evans, R.T.Lewis, H.Siedentop, J.P.Solovej, *Counting eigenvalues using coherent states with an application to Dirac and Schrödinger operators in the semi-classical limit*, Ark.Mat. **34** (1996), 265-283.

[6] V.Ya.Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin- Heidelberg, 1998.

[7] G.D.Raikov, *Eigenvalue asymptotics for the Schrödinger operator in strong constant magnetic fields*, Commun. P.D.E. **23** (1998), 1583-1620.

[8] H.Tamura, *The asymptotic distribution of discrete eigenvalues for Dirac systems*, J.Fac.Sci. Univ. Tokyo Sect. IA Math. **23** (1976), 167-197.

[9] B.Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin-Heidelberg-New York, 1992.