# The splitting of the 2D invariant manifolds along a heteroclinic orbit to saddle-foci for a 3D time-periodic volume-preserving flow.

A. Vieiro

Joint work with A. Murillo (UB)

Universitat de Barcelona

Nonlinear Dynamics: Theory and Applications - Conference named after L.P. Shilnikov

Nizhny Novgorod, May 11th, 2023

# Motivation

The velocity vector field of a fluid defines a 3D flow. Unsteady fluid flows in which the conditions (the velocity, the pressure and the cross-section) change over time, lead to time dependent perturbations of the vector fields.[1]

In this work we consider a periodic forcing of a one-parameter family of 3D vector fields given by the 2-jet of the NF of the volume-preserving elliptic Hopf-zero bifurcation. In particular, we study the consequences of the forcing on the splitting of the 2D separatrices of saddle-foci equilibria and in the dynamics of the chaotic zones.

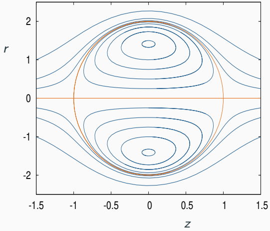
[1] P.Holmes. *Some remarks on chaotic particle paths in time-periodic, three-dimensional swirling flows.* Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983,

M.Feingold, L.P. Kadanoff and O.Piro. *Passive scalars, three-dimensional volume-preserving maps, and chaos.* J. Statist. Phys. 50, 1998.

I. Mezic. *On the geometrical and statistical properties of dynamical systems: Theory and applications* Ph.D. Thesis, California Institute of Technology, 1994.

We consider the integrable family of 3D autonomous flows

$$X_0 := \begin{cases} \dot{x} = y - xz, \\ \dot{y} = -x - yz, \\ \dot{z} = -\varepsilon^2 + z^2 + \frac{1}{2}(x^2 + y^2). \end{cases}$$




$X_0$ is the 2nd order trucation of the NF at the Hopf-zero bifurcation scenario.

It has 2 saddle-foci equilibria $p_\pm = (0, 0, \pm\epsilon)$ with egenvalues $(\mp\varepsilon + i, \mp\varepsilon - i, \pm 2\varepsilon)$.

The 2D invariant manifolds of $p_\pm$ form a 2D invariant sphere foliated by spiralling heteroclinic orbits.

In cylindrical coordinates: $\dot{\theta} = -1, \quad \dot{r} = -rz, \quad \dot{z} = -\varepsilon^2 + z^2 + \frac{r^2}{2}.$

The $(z, r)$-subsystem is Hamiltonian with $H(z, r) = r^2(-\varepsilon^2 + z^2 + \frac{r^2}{4})$.

Note that the angle variable $\theta$ rotates with a frequency 1 while the real part of the eigenvalues at $p_\pm$ is $\mathcal{O}(\varepsilon)$.

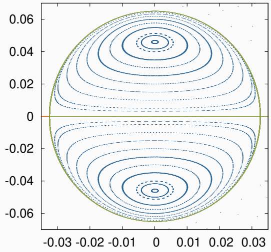
# The periodic forcing

We consider the family of 3D analytic vector field on $\mathbb{R}^3 \times \mathbb{S}^1$,

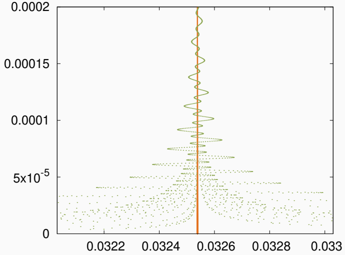
$$X(x,y,z,t) = X_0(x,y,z) + \delta X_1(x,y,z,t),$$

where $X_1 = \left(0, 0, \frac{y(x^2+y^2)}{2(c-y)}\, g(\psi)\right)^T$, $\psi = \omega t + \psi_0$ and $\psi_0 \in [0, 2\pi)$ is an initial phase. We shall consider $\delta$ fixed and study the behavior as $\epsilon \to 0$.

As usual the forcing causes an splitting of the invariant manifolds:




Poincaré section $\{y = 0\}$ for $\varepsilon = 0.03125$ and $\delta = 0.01$. We display the $(x,z)$-coordinates of the iterates.

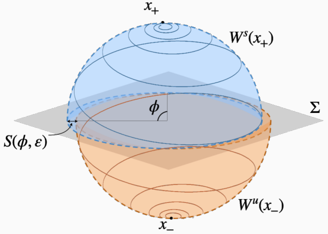



Magnification of the left plot: the propagation of a rotational curve on a local fundamental domain of $W^u(p_-)$ is shown (in green) close to $p_+$.

Autonomous perturbations (e.g. $g(\psi) \equiv$ ctant) breaking the symmetry of the Hopf zero NF have been studied. They lead to exponentially small splitting of separatrices. The splitting function[2,3] measured in a suitable Poincaré section $\Sigma$ is

$$S(\phi, \varepsilon) = r^u(\phi, \varepsilon) - r^s(\phi, \varepsilon) = F(\phi, \log \varepsilon)\, \varepsilon^{-3} \exp\left(-\frac{\pi}{2\varepsilon}\right),$$

where $F(\phi, \log \varepsilon) = C_1^* \cos(\phi - L_0 \log \varepsilon) + C_2^* \sin(\phi - L_0 \log \varepsilon) + O(|\log \varepsilon|^{-1})$, with $C_1^*$, $C_2^*$ and $L_0$ real constants.




[2] Dumortier F., Ibáñez S., Kokubu H. and Simó C. About the unfolding of a Hopf-zero singularity, *Discrete Contin Dyn Syst* **33**: 4435-4471, 2013.

[3] Baldomá I., Ibáñez S. and Seara T.M. Hopf-Zero singularities truly unfold chaos, *Commun Nonlinear Sci Numer Simulat* **84**, 2020.

## Non-autonomous forcing

When a non-autonomous forcing is considered the two frequencies 1 and $\omega$ can interact. The Fourier expansion of the perturbation in the $(\theta, \psi)$ angles restricted along the heteroclinic orbits of the unperturbed system can give terms with all possible frequency combinations (small divisors). This makes the description of the asymtotic behavior of the splitting much involved. For illustrations we shall consider

$$g(\psi) = \frac{1}{d - \cos(\psi)}, \qquad \psi = \omega t + \psi_0,$$

which is a $\frac{2\pi}{\omega}$-periodic (in $t$) forcing.

As usual[2] the arithmetic properties of $\omega$ play a role in the splitting behavior. For concreteness we consider $\omega = \sqrt{2}$ below.

[2] E. Fontich, C. Simó, and A. Vieiro. Splitting of the separatrices after a Hamiltonian–Hopf bifurcation under periodic forcing. *Nonlinearity*, **32**(4):1440–1493, 2019.

## The Melnikov function

Given $\delta > 0$, the intersection of $W^{u,s}(p_\pm)$ with $\{z = 0\}$ can be parameterized by $p^{u,s}(\delta) = p^{u,s}(\theta_0, \psi_0, \varepsilon, \delta)$. The distance

$$d(p^u(\delta), p^s(\delta)) = H(p^u(\delta)) - H(p^s(\delta)) = \delta M(\theta_0, \psi_0, \varepsilon) + O(\delta^2),$$

is given by

$$M(\theta_0, \psi_0, \varepsilon) = \sum_{m_1 \geq 0} \sum_{m_2 \in \mathbb{Z}} C_{m_1,m_2} f(\theta_0, \psi_0),$$

where $f(\theta_0, \psi_0) = \begin{cases} \cos(m_1\theta_0 + m_2\psi_0), & m_1 \text{ odd}, \\ \sin(m_1\theta_0 + m_2\psi_0), & m_1 \text{ even}, \end{cases}$ and

$$|C_{m_1,m_2}| = \frac{2^6 \pi \rho_d^{m_2}}{\sqrt{d^2 - 1} c^{m_1}} \exp\left(\frac{-s\pi}{2\varepsilon}\right) \sum_{i \geq 0} \frac{(m_1 + 2i)!\ \Pi_{m_1+2i+4}(s)}{c^{2i}(m_1 + i)!(m_1 + 2i + 4)!i!},$$

where $s = |m_2\omega - m_1|$, $\rho_d = (d + \sqrt{d^2 - 1})^{-1}$ and

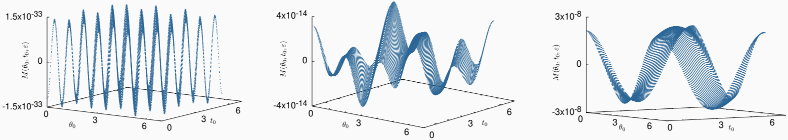
$$\Pi_m(s) = (s^2 + \varepsilon^2(m-2)^2)\Pi_{m-2}(s), \qquad \Pi_1 = s, \quad \Pi_0 = 1.$$

## 1st order in $\delta$ splitting function = Melnikov function?

The validity of the Melnikov approximation is (a priori) not justified in this setting. Note also that the 2nd and higher order terms in $\delta$ of the expansion of the Melnikov function are also exponentially small in $\epsilon$.

However:

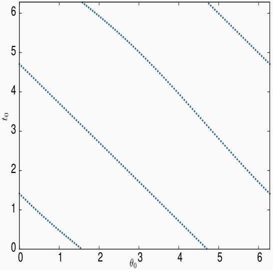
For $\delta$ fixed and small and $\varepsilon \searrow 0$, we can directly compute the invariant manifolds and provide evidence that the Melnikov function gives a good approximation of the splitting function.



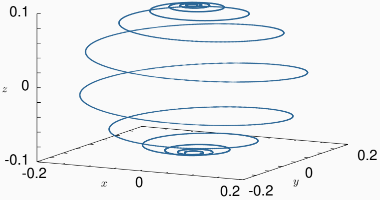

$\varepsilon = 0.005$ $\varepsilon = 0.0321$ $\varepsilon = 0.1$

Splitting function w.r.t. $(\theta_0, \psi_0)$ for a fixed $\delta = 0.01$ and $\omega = \sqrt{2}$.

The points $(\theta_0, \psi_0)$ for which the Melnikov function vanishes correspond to heteroclinic orbits of the perturbed system. For $\varepsilon = 0.1, \delta = 0.01$:




Continuum of heteroclinic orbits.



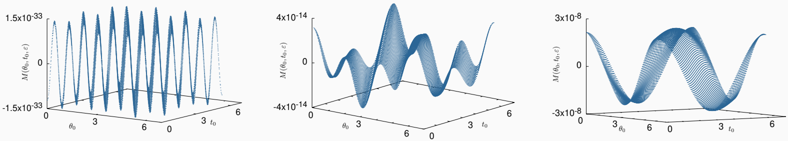

Heteroclinic orbit for $(\psi_0, \theta_0) = (4.47, 3.52)$.

---

H. E. Lomelí and R. Ramírez-Ros. Separatrix Splitting in 3D Volume-Preserving Maps. *Siam J. Applied Dynamical Systems*,**7**: 1527–1557,2008.

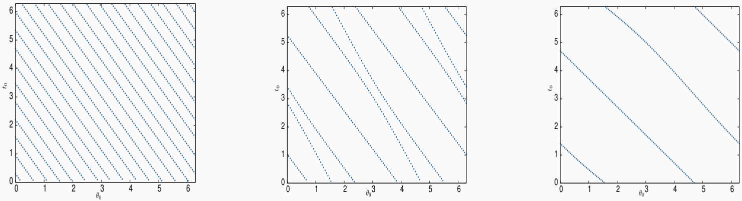
# The continuum of heteroclinic orbits

Splitting function w.r.t. $(\theta_0, \psi_0)$ for a fixed $\delta = 0.01$ and $\omega = \sqrt{2}$.




$\varepsilon = 0.005$ $\quad$ $\varepsilon = 0.0321$ $\quad$ $\varepsilon = 0.1$

$m_1/m_2 = 7/5$ $\quad$ $m_1/m_2 = 3/2$ $\quad$ $m_1/m_2 = 1/1$




Continuum of heteroclinic orbits for $\omega = \sqrt{2}$, i.e. $M(\theta_0, \psi_0, \varepsilon) = 0$.

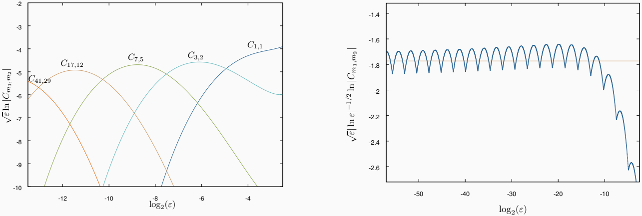
Amplitude of the Fourier modes of $M(\theta_0, \psi_0, \varepsilon)$,

$$|C_{m_1,m_2}| \sim \frac{\rho_d^{m_2}}{c^{m_1}(m_1+4)!}\Pi_{m_1+4}(s)\exp\left(\frac{-s\pi}{2\varepsilon}\right),$$

where $s = |m_2\omega - m_1|$ and $\Pi_m(s) = (s^2 + \varepsilon^2(m-2)^2)\Pi_{m-2}(s)$, $\Pi_1 = s$, $\Pi_0 = 1$.

For constant type $\omega$ the maximum value of $|C_{m_1,m_2}|$ is achieved for $s(\varepsilon) \approx \sqrt{\frac{2}{k\pi}}\sqrt{\varepsilon}|\ln\varepsilon|^{1/2} + O(\sqrt{\varepsilon})$, and then

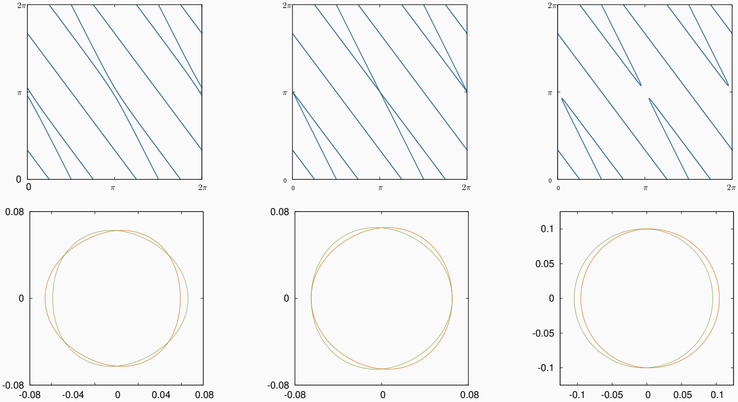
$$|C_{m_1,m_2}| \sim \exp\left(\frac{-\sqrt{\pi}|\ln\varepsilon|^{1/2}}{\sqrt{\varepsilon}}\right).$$




We display $\sqrt{\varepsilon}\ln|C_{m_1,m_2}|$ (left) and $\sqrt{\varepsilon}|\ln\varepsilon|^{-1/2}\ln|C_{m_1,m_2}|$ (right) for some best approximants $m_1/m_2$ of $\omega = \sqrt{2}$ as a function $\log_2(\varepsilon)$.

# Changes in the topology of the nodal lines ($\delta = 0.01$)

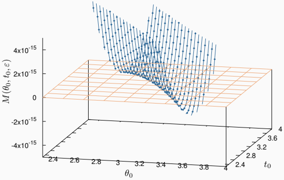
When $\varepsilon$ varies, there are changes in the topology of the nodal lines of the Melnikov function.



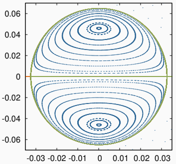

We display the nodal lines of the Melnikov function for $\varepsilon < \varepsilon_*$, for $\varepsilon = 0.03253 \approx: \varepsilon_*$ and $\varepsilon > \varepsilon_*$ (from left to right). On $z = 0$ (the "equator"), we display $(x, y)$-projection of the invariant manifolds for $\psi_0 = \pi$ (the relative distance is magnified by a suitable factor).

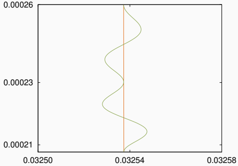
# Saddle-type tangencies between the manifolds

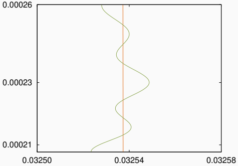
Each change of the topology of the nodal lines corresponds to a change of the dominant harmonic of the Melnikov function and to a quadratic tangency between the 2D invariant manifolds (locally a hyperbolic paraboloid, that is at $\varepsilon = \varepsilon_*$ one has a saddle critical point).




For $\epsilon = \epsilon_*$ we consider the image of points on the invariant manifolds on $\{z = 0\} \cap \{\psi_0 = \pi\}$ (center) and $\{z = 0\} \cap \{\psi_0 = 0\}$ (right). We display the $(x, z)$-coordinates at a suitable crossing with the Poincaré section $\{y = 0\}$ near the saddle-focus point $p_+$.



## Remarks

The example provided is of the lowest possible dimension (1 action, 2 angles) to exhibit the phenomena of interaction of two frequencies.

The same phenomena of splitting of separatrices is observed for discrete 3D volume-preserving maps (for example, take a Poincaré section of the system considered). Together with M.Gonchenko and J.C.Tatjer we aim to investigate dynamics near tangencies (of both types) of two dimensional manifolds in 3D volume-preserving maps.

The splitting of separatrices creates regions with chaotic dynamics. Such chaotic dynamics can be studied by using a separatrix return map to a domain (of width the order of the splitting) around the invariant manifolds.

As far as we are aware there are no systematic studies of the consequences of the interaction of frequencies in the asymptotic behavior of the splitting of separatrices in the chaotic region. $\rightsquigarrow$ Separatrix map 3D

## A separatrix map model: preliminary ideas (in progress)

We are working on the derivation of a formal separatrix map to study the effect of the described splitting of separatrices in the dynamics within chaotic zones.

Our results show that, for some ranges of the parameter $\varepsilon$, two harmonics (corresponding to consecutive best approximants of $\omega$) need to be considered to provide the dominant terms of the splitting function.

We express the return map in variables $(h, \theta, \psi)$, where $h$ stands for the value of the 2D Hamiltonian of the unperturbed system. Under some assumptions and a rescaling of $h$, a suitable return map model to illustrate the transition $(m_1, m_2) \to (n_1, n_2)$ of dominant harmonics is

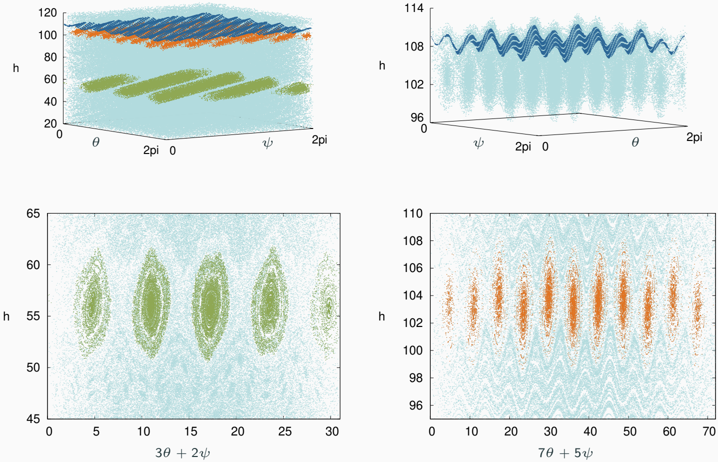
$$\begin{aligned}
\bar{h} &= h + \cos(m_1\bar{\theta} + m_2\bar{\psi}) + s_1^p \cos(n_1\bar{\theta} + n_2\bar{\psi}), \\
\bar{\theta} &= \theta + \omega t_v(h) + a, \qquad \pmod{2\pi} \\
\bar{\psi} &= z + t_v(h) + a, \qquad \pmod{2\pi}
\end{aligned}$$

where

$$t_v(h) = -\log(|h|)/(2\epsilon) \qquad s_1 = |m_1 - m_2\omega|, \quad s_2 = |n_1 - n_2\omega|,$$

$$p = p(s_1, s_2, \epsilon) = (s_2 - s_1)(\epsilon - 1)\pi/(2\epsilon), \qquad a \text{ constant}$$

## SM model plots $\omega = \sqrt{2}$, $m_1 = 3$, $m_2 = 2$, $n_1 = 7$, $n_2 = 5$




Different resonances (aligned according to the resonant lines) are observed.
Chaotic zone is $\mathcal{O}(\exp(-c/\sqrt{\epsilon}))$ (the variable $h$ is scaled accordingly).

## Final comments

Many other things can be obtained from the previous results on splitting and the separatrix map, for example...

## Final comments

Many other things can be obtained from the previous results on splitting and the separatrix map, for example...

... a traditional 3D KAM cake with resonances and a cover of 2D invariant tori!

**Happy birthday Sergey!!**
**Thank you for you attention!**