Splitting of the separatrices after a Hamiltonian-Hopf bifurcation under periodic forcing.

Perspectives in Hamiltonian Dynamics

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The problem: general setting

Let $H_0(\nu) = H_0(\nu, x_1, x_2, y_1, y_2)$ be a one-parameter ν-family of 2-dof Hamiltonian systems such that

1. the origin is a fixed point for all ν,
2. at $\nu = 0$ the origin suffers a Hamiltonian-Hopf bifurcation, and
3. for $\nu > 0$ the invariant manifolds of the origin (complex unstable) form a "homoclinic 2-dimensional figure-eight".

We consider

a periodic in time forcing $H = H_0(\nu) + \epsilon H_1$ (ϵ small and fixed) on the family (hence 2+1/2 dof Hamiltonian system).

Our goal is

to describe the asymptotic behaviour (when $\nu \to 0$) of the splitting of the invariant manifolds.

The concrete system

Concretely, we consider the system

$$H(x_1, x_2, y_1, y_2, t) = H_0(x_1, x_2, y_1, y_2) + \epsilon H_1(x_1, x_2, y_1, y_2, t),$$

where

$$H_0 = x_1 y_2 - x_2 y_1 + \nu \left(\frac{x_1^2 + x_2^2}{2} + \frac{y_1^2 + y_2^2}{2} \left(-1 + \frac{y_1^2 + y_2^2}{2} \right) \right),$$

and

$$H_1 = \frac{y_1^5}{(d - y_1)(c - \cos(\theta))}, \quad \theta = \gamma t + \theta_0.$$

1. We shall fix concrete values of c, d, γ and ϵ.
2. $\nu > 0$ is a perturbative parameter.
3. The parameter $\theta_0 \in [0, 2\pi)$ is the initial time phase.
4. Note that H_1 contains all powers $y_1^k, k > 4$ and all harmonics in θ.

Why this concrete system? H_0 *?*

Consider a 1-param. family of 2-dof Hamiltonians H_δ undergoing a Hamiltonian-Hopf bifurcation (at the origin).

Assume: for $\delta > 0$ elliptic-elliptic, $\delta < 0$ complex-saddle.

The NF analysis of the HH bifurcation leads to the so-called **Sokolskii NF**:

$$\mathrm{NF}(H_\delta) = \omega\Gamma_1 + \Gamma_2 + \sum_{\substack{k,l,j\geq 0\\ k+l\geq 2}} a_{k,l,j}\,\Gamma_1^k\,\Gamma_3^l\,\delta^j, \qquad \longleftarrow \text{ formal}$$

where

$$\Gamma_1 = x_1y_2 - x_2y_1, \quad \Gamma_2 = (x_1^2 + x_2^2)/2 \quad \text{and} \quad \Gamma_3 = (y_1^2 + y_2^2)/2.$$

Γ_1 is a (formal) first integral, hence $W^{u/s}(\mathbf{0}) = \{\Gamma_1 = 0\} \cap \{\mathrm{NF}(H_\delta) = 0\}$.

- If $a_{0,2,0} > 0$ they bound a finite domain of size $\Gamma_2 = \mathcal{O}(\delta^2), \Gamma_3 = \mathcal{O}(\delta)$.
- If $a_{0,2,0} < 0$ they are unbounded.

The unperturbed model: H_0

We consider the bounded case.

Introducing $\delta = -\nu^2$, and rescaling $x_i = \nu^2 \tilde{x}_i, \omega y_i = \nu\, \tilde{y}_i, i = 1, 2,$ $\omega t = \tilde{t}$, one has (skipping ˜ from the new variables)

$$\boxed{\mathrm{NF}(H_\delta) = \Gamma_1 + \nu \left(\Gamma_2 + a\Gamma_3 + \eta\Gamma_3^2\right) + \mathcal{O}(\nu^2)}$$

where $a = -a_{0,1,1}/\omega^2$ and $\eta = a_{0,2,0}/\omega^4$.

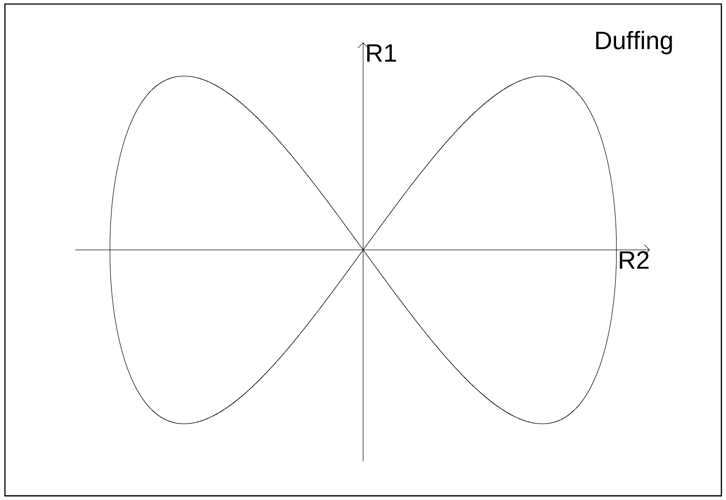
Taking $a = -1, \eta = 1$, and truncating we obtain the unperturbed integrable system considered:

$$\begin{aligned} H_0 =& x_1 y_2 - x_2 y_1 + \nu \left(\frac{x_1^2 + x_2^2}{2} + \frac{y_1^2 + y_2^2}{2} \left(-1 + \frac{y_1^2 + y_2^2}{2} \right) \right) \\ =& \Gamma_1 + \nu(\Gamma_2 - \Gamma_3 + \Gamma_3^2). \end{aligned}$$

Then, $G_1 = \Gamma_1$ and $G_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2$ are independent first integrals.

Geometry of the invariant manifolds for H_0

In polar coord $x_1 + i x_2 = R_1 e^{i\psi_1}$, $y_1 + i y_2 = R_2 e^{i\psi_2}$ the restriction to (R_1, R_2)-components is a Duffing Hamiltonian system.



On $W^{u/s}(\mathbf{0})$ one has $\psi_1 = \psi_2 \pm \pi$, $\psi_2 = t + \psi_0$. The 2-dimensional homoclinic surface is foliated by homoclinic orbits $(x_1(t), x_2(t), y_1(t), y_2(t))$ given by

$$x_1(t) + i x_2(t) = -R_1(t) e^{i\psi(t)}, \; y_1(t) + i y_2(t) = R_2(t) e^{i\psi(t)},$$

being $\psi(t) = t + \psi_0$, $R_1(t) = \sqrt{2}\,\mathrm{sech}(\nu t)\tanh(\nu t)$, and $R_2(t) = \sqrt{2}\,\mathrm{sech}(\nu t)$.

Periodic forcing: ϵH_1

We add to H_0 the periodic perturbation $\epsilon H_1 = \epsilon g(y_1) f(\theta)$ where

$$g(y_1) = y_1^5 (d - y_1)^{-1}, \qquad f(\theta) = (c - \cos(\gamma t + \theta_0))^{-1}.$$

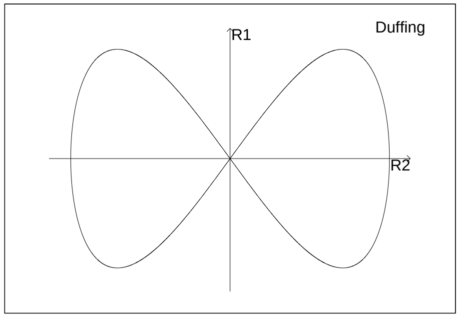
Remarks:

1. Restricted to the unperturbed $W^{u/s}(\mathbf{0})$, y_1 becomes 1-periodic in t.
2. $f(\theta)$ periodic in t with frequency $\gamma \Rightarrow$ If $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ then quasi-periodic!

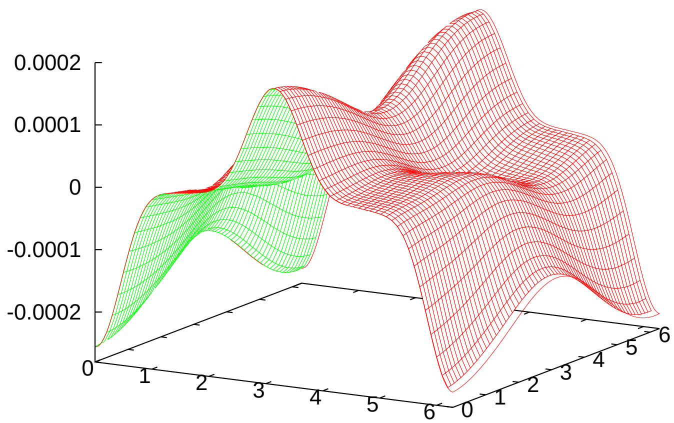
We consider for numerical simulations $c = 5$, $d = 7$, and $\epsilon = 10^{-3}$.
Also $\gamma = \gamma_0 = (\sqrt{5} - 1)/2$ (later other values of γ).

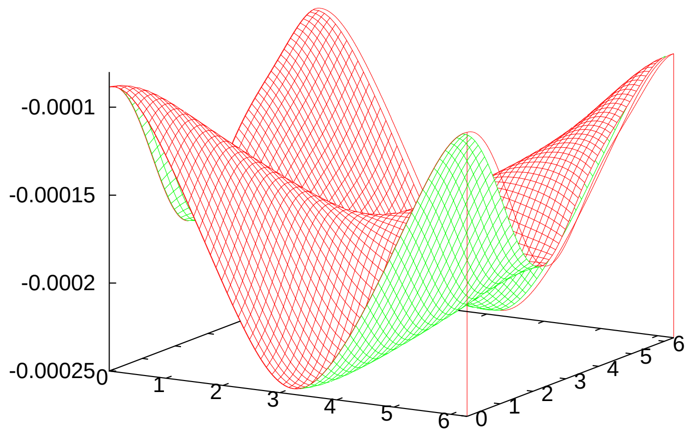
Recall that (ψ_0, θ_0) are initial conditions on a fundamental domain (torus $\mathcal{T}$) of $W^{u/s}(\mathbf{0})$. Also recall that ν is a small parameter (included in H_0).

The invariant manifolds $W^{u/s}(0)$



We express $H_0 = G_1 + \nu G_2,\ G_1 = \Gamma_1,\ G_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2,$ and we consider the Poincaré section $\Sigma = \max(R_2)$.
The values are represented as functions of (ψ, θ).

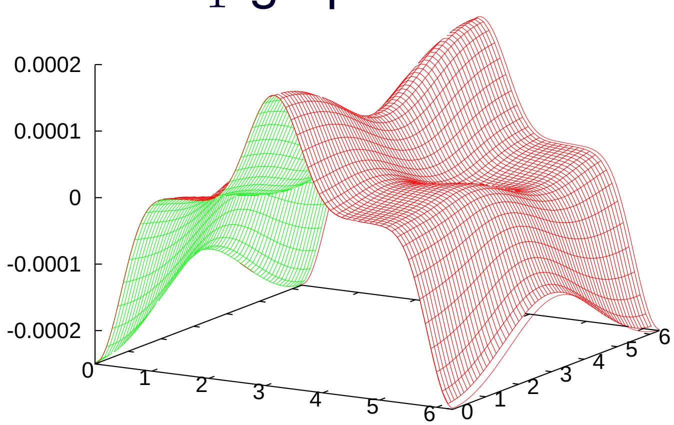


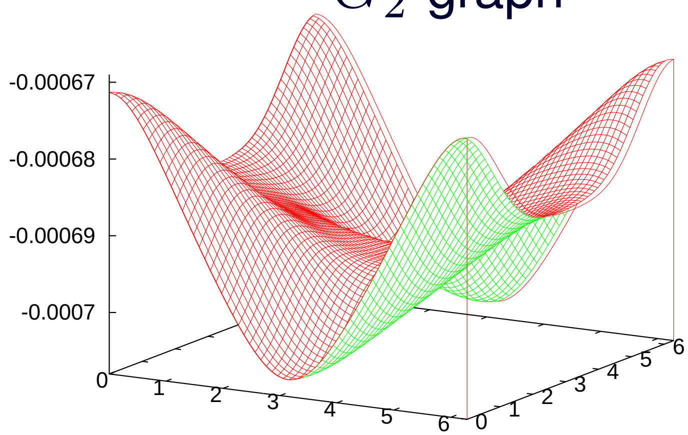


$$\nu = 2^{-4}$$

G_1-graph

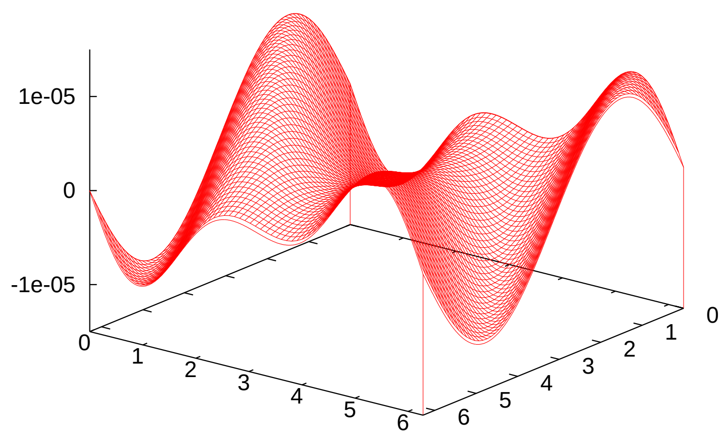
G_2-graph



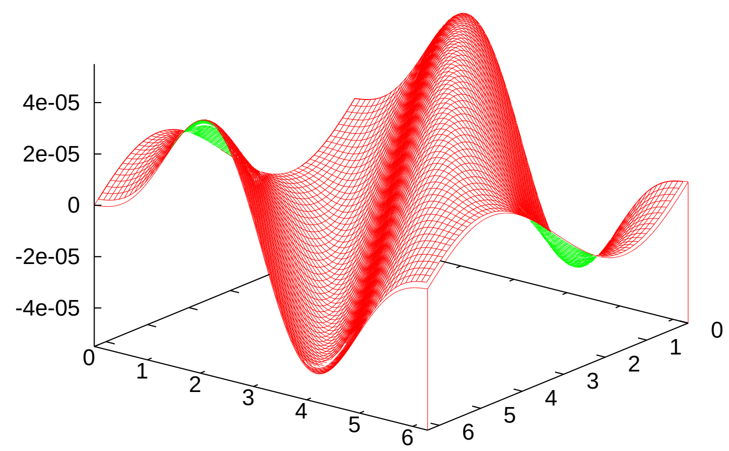


$$\nu = 2^{-6}$$

The splitting of the invariant manifolds

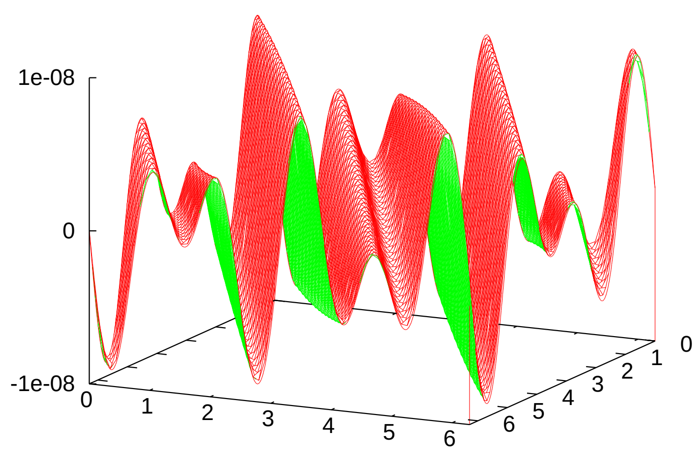


ΔG_1 (1,1)

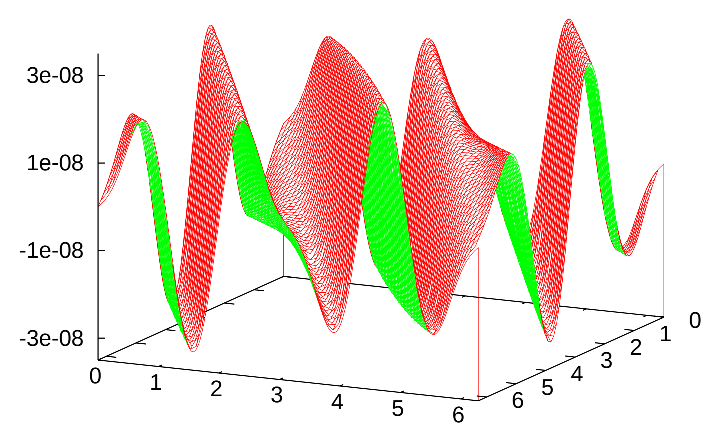


ΔG_2 (1,1)

$\nu = 2^{-4}$



ΔG_1 (3,5)



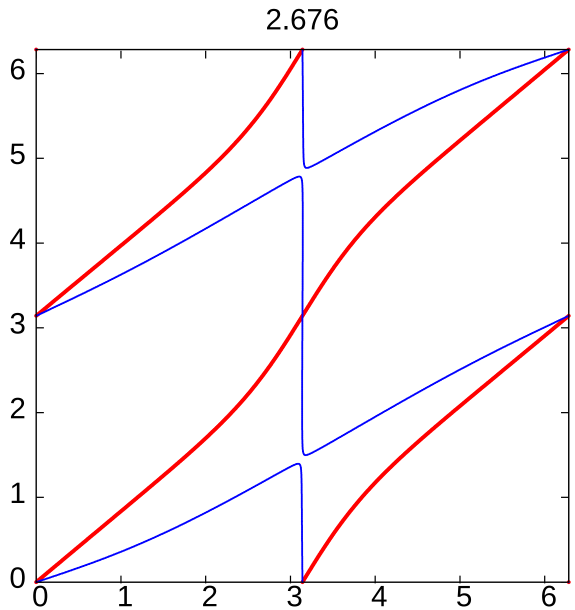
ΔG_2 (2,3)

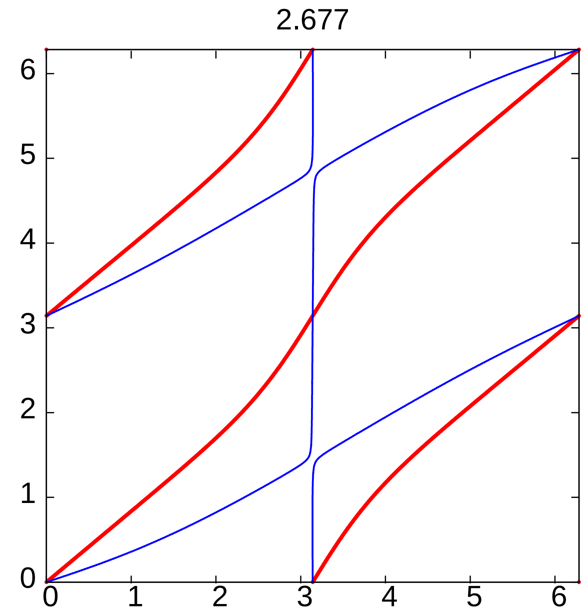
$\nu = 2^{-6}$

Remarks on the previous computations

1. We propagate a set $\{\psi_{0,k}, \theta_{0,j}\}$, $0 \leq k, j \leq 512$, of initial points in the fundamental torus $\mathcal{T}$ (i.e. a total number of 2^{18} initial conditions) up to reach the Poincaré section Σ.
2. The numerical integration is performed using an ad-hoc implemented Taylor time-stepper scheme with quadruple precision.
3. The propagation of $\mathcal{T}$ up to Σ gives a 2D torus $\mathcal{T}_\Sigma$. The invariant manifolds $W^{u/s}(\mathbf{0})$ in $\mathbb{R}^4$ are defined by the G_1 and the G_2-graphs over $\mathcal{T}_\Sigma$.
4. To compute the difference (i.e. the splitting) between $W^u(\mathbf{0})$ and $W^s(\mathbf{0})$ we need to compare them at the same points of $\mathcal{T}_\Sigma$. Hence, we select a mesh of angles ψ and θ within $\mathcal{T}_\Sigma$, and refine the initial conditions in $\mathcal{T}$ using a Newton method.

Nodal lines: changes of the dominant harmonic

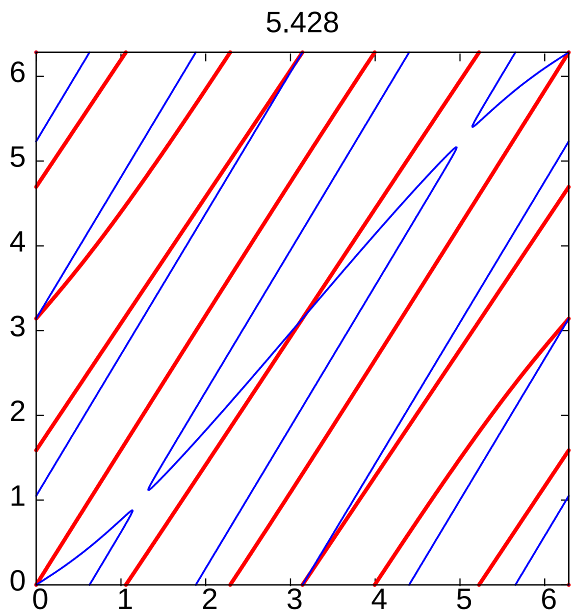


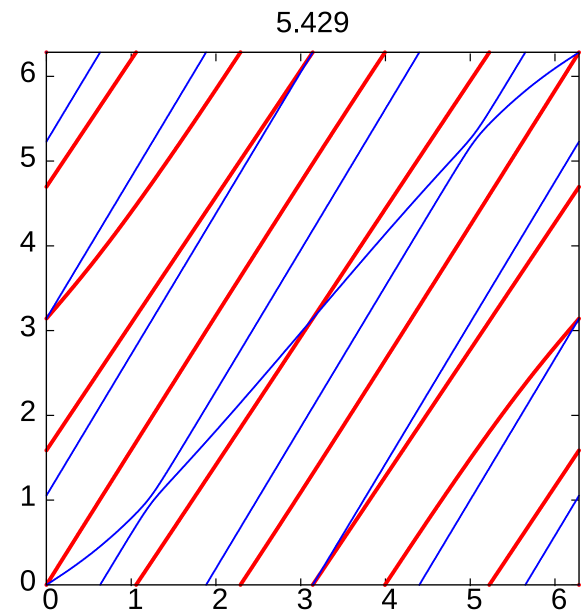


(1,1), (1,0)

↘

(1,1), (1,1)





(2,3), (1,2)

↘

(2,3), (2,3)

Change of dominant harmonics

$-\log_2 \nu_+$	$-\log_2 \nu_-$	Change of the dom harm of the G_1, G_2-splittings
2.443	2.444	(1,0), (1,0) $\longrightarrow$ (1,1), (1,0)
2.676	2.677	(1,1), (1,0) $\longrightarrow$ (1,1), (1,1)
4.112	4.113	(1,1), (1,1) $\longrightarrow$ (1,2), (1,1)
4.300	4.301	(1,2), (1,1) $\longrightarrow$ (1,2), (1,2)
5.133	5.134	(1,2), (1,2) $\longrightarrow$ (2,3),(1,2)
5.428	5.429	(2,3), (1,2) $\longrightarrow$ (2,3), (2,3)
5.971	5.972	(2,3),(2,3) $\longrightarrow$ (3,5), (2,3)
6.234	6.235	(2,3), (2,3) $\longrightarrow$ (3,5), (2,3)

Table 1: Changes in the dominant harmonic of the G_1 splitting function and the G_2 splitting function. The change takes place for $\nu \in (\nu_-, \nu_+)$.

The Melnikov integral

For simplicity, we discuss on the G_1-splitting (similar for the G_2-splitting).

Recall that $H_1 = g(y_1) f(\theta)$ where

$g(y_1) = y_1^5 (d - y_1)^{-1} \rightsquigarrow g'(y_1) = \sum_{k \geq 0} d_k y_1^{4+k}$,

$f(\theta) = (c - \cos(\theta))^{-1} = \sum_{j \geq 0} c_j \cos(j\theta)$.

The P-M function:

If $\zeta^0(s)$ is a solution of the system when $\epsilon = 0$, then the distance

$$G_1^u(\psi_0, \theta_0) - G_1^s(\psi_0, \theta_0) = \Delta G_1 + \mathcal{O}(\epsilon^2),$$

is given by

$$\Delta G_1 = \epsilon \int_{-\infty}^{\infty} \{G_1, H_1\} \circ \zeta^0(s)\, ds + O(\epsilon^2)$$

$$= 4\epsilon \int_{-\infty}^{\infty} \sin(t + \psi_0)\, f(\gamma t + \theta_0) \sum_{k \geq 0} \frac{\sqrt{2^{k+1}}\, d_k\, (\cos(t + \psi_0))^{4+k}}{(\cosh(\nu t))^{5+k}} dt.$$

Recall that on the unperturbed separatrices $\psi = t + \psi_0$, $\theta = \gamma t + \theta_0$, $(\psi_0, \theta_0) \in \mathcal{T}$.

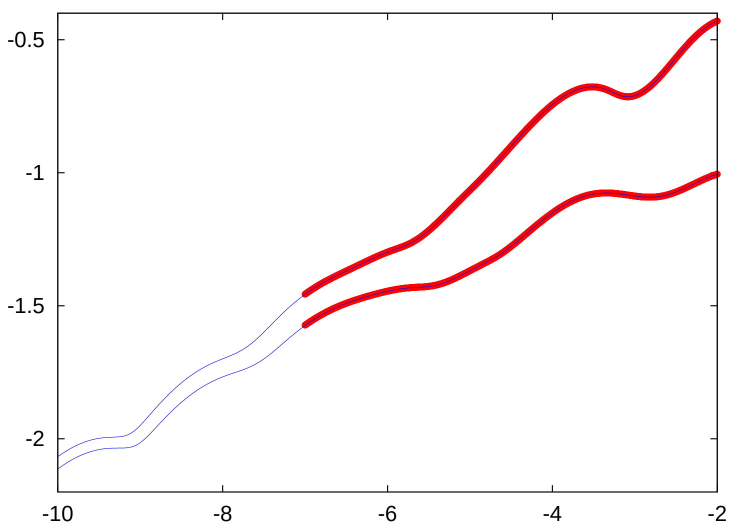
Comparison numerics/symbolic evaluation

After some algebra one obtains

$$\Delta G_1 = \epsilon \sum_{j\geq 0} c_j \sum_{k\geq 0} 2^{\frac{3+k}{2}} d_k \sum_{0\leq 2i\leq 4+k} b_{4+k,i} \sum_{l=\pm 1} I_1 \sin((k+5-2i)\psi_0 + lj\theta_0)$$

$$= \epsilon \sum_{m_1\geq 0} \sum_{m_2\in\mathbb{Z}} C^{(1)}_{m_1,m_2} \sin(m_1\psi_0 - m_2\theta_0),$$ where

$$I_1 = I_1(k+5-2i+lj\gamma, \nu, k+5), \quad I_1(s,\nu,n) = \int_{\mathbb{R}} \frac{\cos(st)}{(\cosh(\nu t))^n}\, dt, \quad b_{m,i} = \frac{m+1-2i}{2^m(m+1)} \begin{pmatrix} m+1 \\ i \end{pmatrix}$$



We represent $\log(\Delta G_i/\epsilon)\sqrt{\nu}$, for $i=1$ (bottom) and $i=2$ (top), as a function of $\log_2(\nu)$.

Red: Direct numerical computations.

Blue: Sum of the significant terms of the Melnikov series.

Main theoretical result

For the system $H = H_0 + \epsilon H_1$ under consideration, let us assume that

$$\epsilon > 0, \quad c > 1,\ d > \sqrt{2}, \quad \gamma \in \mathbb{R} \setminus \mathbb{Q} \quad \text{and} \quad \nu < \nu_M \ll 1.$$

Let m_1/m_2 be an approximant of γ, and let $c_s \in \mathbb{R}$ be the constant such that

$$c_s m_1 |m_1 - \gamma m_2| = 1.$$

Thm. There exists a “universal” (almost independent of γ) function $\psi_1(L)$ s.t. the contribution of the harmonic associated to m_1/m_2 to the splitting satisfies

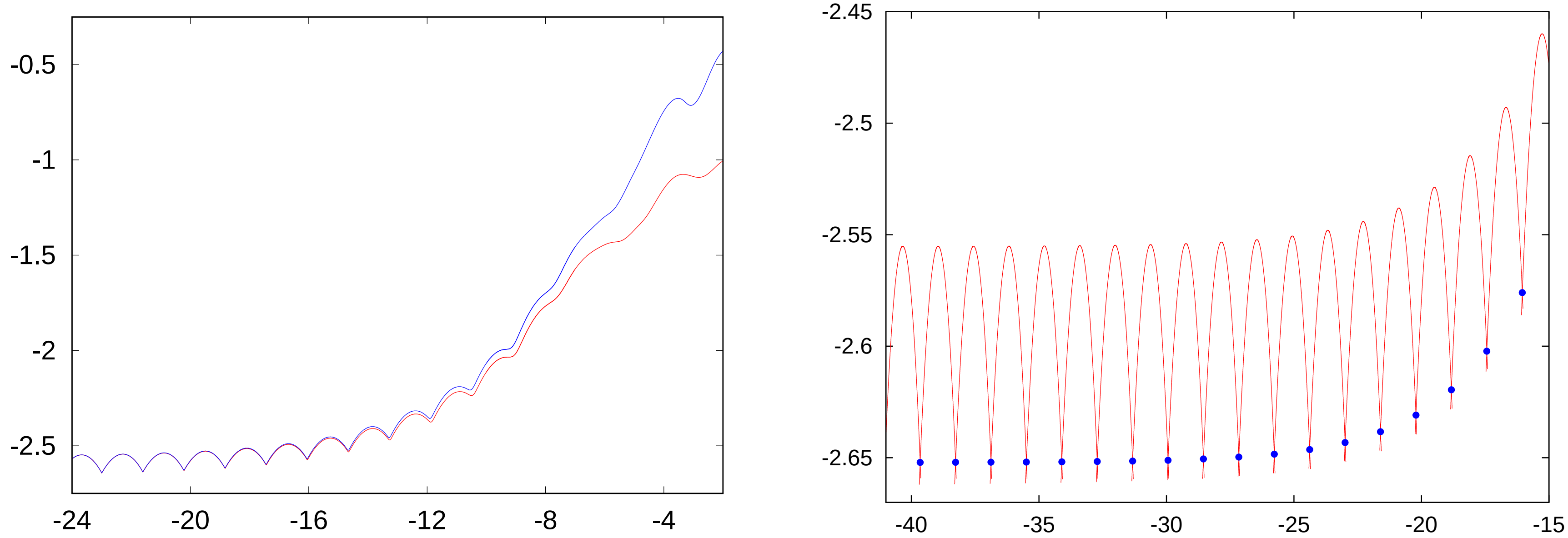
$$\psi_i(L)|_{L=c_s \nu m_1^2} \approx \sqrt{c_s \nu} \log |C^{(i)}_{m_1,m_2}|, \ \text{ when } \nu \to 0,$$

where $\Psi_2(L) = \Psi_1(L) - \sqrt{L} \log L/m_1$, $\Psi_i(L) \leq \Psi_M \approx -4.860298$.

In particular, if m_1/m_2 corresponds to a dominant HBA of ΔG_1 (resp. ΔG_2) for $\nu \in (\nu_0, \nu_1)$, $\nu_0, \nu_1 \ll 1$, then

$$\Delta G_i \approx \exp\left(\psi_i(L)|_{L=\nu m_1^2 c_s}/\sqrt{\nu}\right), \quad i = 1, 2,$$

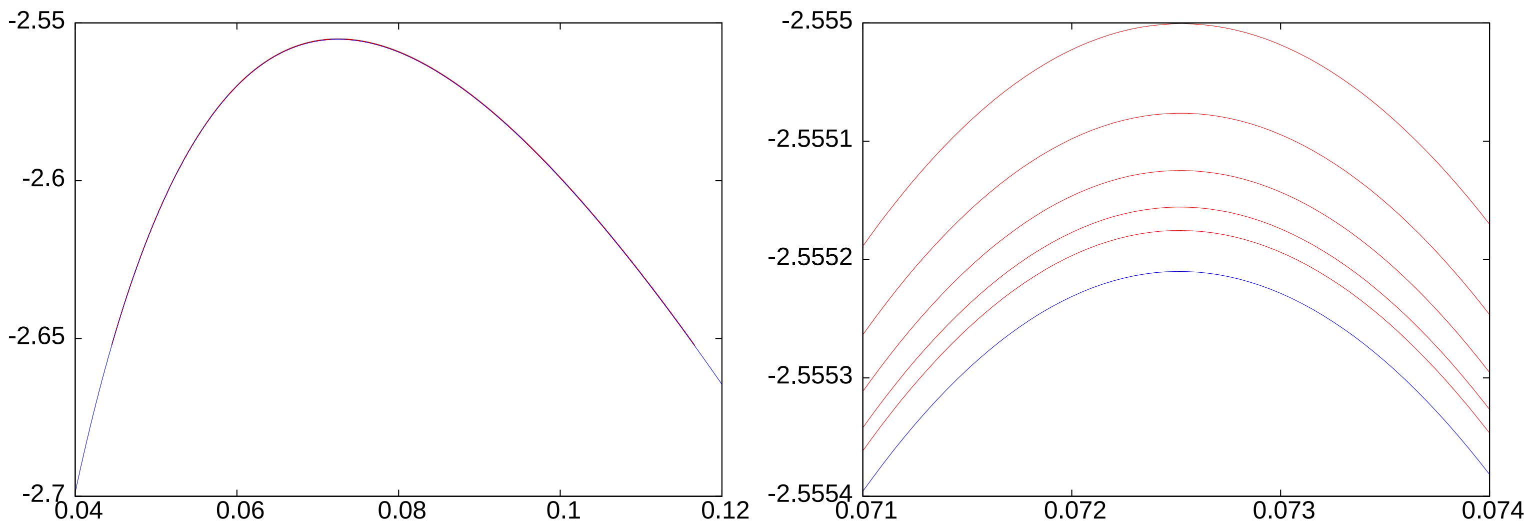
Changes of the dominant harmonic



For $\gamma = (\sqrt{5}-1)/2$, $\epsilon = 10^{-4}$ we represent $\sqrt{\nu}\log|C^{(1)}_{m_1,m_2}/\epsilon|$ as a function of $\log_2(\nu)$. The points correspond to the values ν_j where changes the dominant harmonic. As expected, dominant harmonics are associated to BA: from $m_1 = F_j \to F_{j+1}$, where $\{F_j\}_j$ denotes the Fibonacci sequence. The rightmost change corresponds to $m_1 = 55 \to m_1 = 89$, while the leftmost to $m_1 = 196418 \to m_1 = 317811$.

The function $\psi_1(L)$

We have an explicit expression of $\psi_1(L)$. For $\gamma = (\sqrt{5}-1)/2$ one has $c_s \approx \sqrt{5}(1+\gamma)$. Denote by $\tilde{L} = L/c_s$.



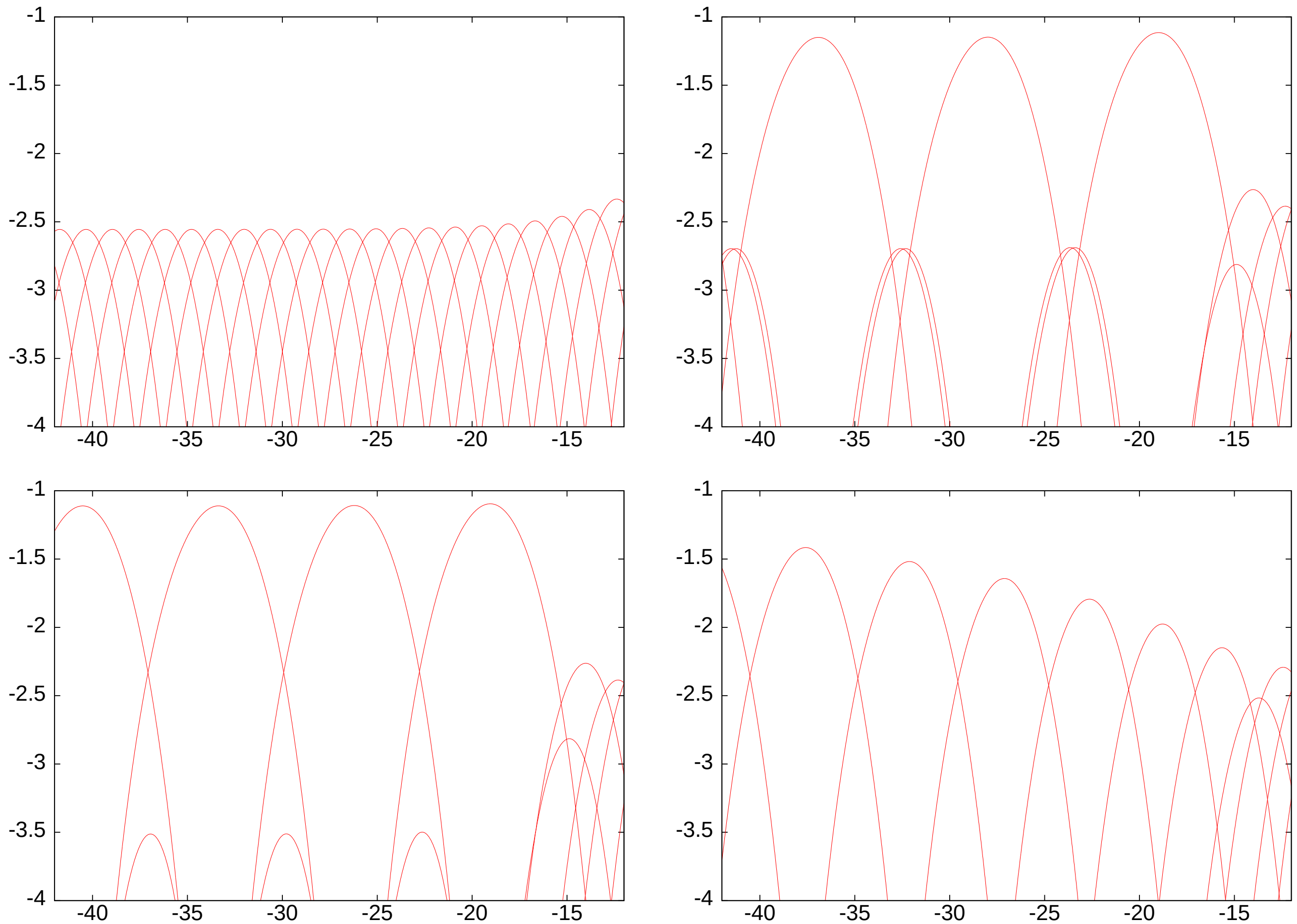
Left: Five leftmost picks of the previous fig. as a function of $\tilde{L}$ (in red).

The function $\psi_1(\tilde{L})$ is diplayed in blue.

Right: Magnification of the central zone of the left plot.

The red curves tend to $\psi_1(\tilde{L})$ as ν decreases (and m_1 increases).

Other frequencies: BA and hidden BA (HBA)



We display $\sqrt{\nu}(\log(\Delta G_1)/\epsilon)$ as a function of $\log_2(\nu)$.

Top left : $\gamma_0 = (\sqrt{5}-1)/2 = [0; 1, 1, 1, 1, 1, ...] \approx 0.618033988749894.$

Top right : $\gamma_1 = [0; 10\times 1, 1, 10, 1, 1, 10, 1, 1, 10, 1, ...] \approx 0.618051226819253.$

Bottom left : $\gamma_2 = [0; 10\times 1, 1, 10, 1, 10, 1, 10, 1, 10...] \approx 0.618051374461158.$

Bottom right: $\gamma_3 = [0; 10\times 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...] \approx 0.618020663293438.$

Hidden HBA: questions and assumptions

As said it is reasonable to expect that BA are dominant. But...

1. γ_2 has some hidden BA harmonics (HBA) [Delshams-Gutierrez-Gonchenko 2014]

   Q: Why some BA never dominate for any ν? Which conditions satisfy?

2. all frequencies $\gamma_i,\ i = 0, 1, 2, 3$, shown before are rather "special".

   Q: What is expected for "typical" (full measure set) frequency γ.

Let us assume that (our system satisfies these assumptions):

- The perturbation is the product of two functions $f(x_1, x_2, y_1, y_2)$ and $g(\theta)$, denote by $\mathcal{P}_1(t, \psi)$ and $\mathcal{P}_2(\theta)$ their contribution to the P-M integral.
- The homoclinic conections tend to zero when $t \to \pm\infty$ as $\operatorname{sech}(\nu t)$.
- $\mathcal{P}_1(t, \psi)$ is of the form $\sum A_j(t) \sin(j\psi)$, $\psi = t + \psi_0$, where A_j depend on powers of $\operatorname{sech}(t)$ and $\|A_j\| \sim \exp(-j\rho_1)$, $\rho_1 > 0$,
- $\mathcal{P}_2(\theta)$ is of the form $B \sum_{j\geq 1} \exp(-j\rho_2) \cos(j\theta)$, $\theta = \gamma t + \theta_0$, $\rho_2 > 0$.

Contribution of HBA

Under previous assumptions, one has that minus the logarithm of the contribution of the harmonic related to the BA N_k/D_k to the P-M function is

$$T(\nu, D_k) \approx D_k + s_k/\nu,$$

where $s_k = |N_k - \gamma D_k|$ and where we have approximated $N_k = \gamma D_k + \mathcal{O}(D_k^{-2})$. The role of CFE appears as

$$s_k^{-1} = D_k \left(c_k^+ + 1/c_k^-\right), \; c_k^+ = [q_{k+1}; q_{k+2}, \ldots], \; c_k^- = [q_k; q_{k-1}, \ldots, q_1].$$

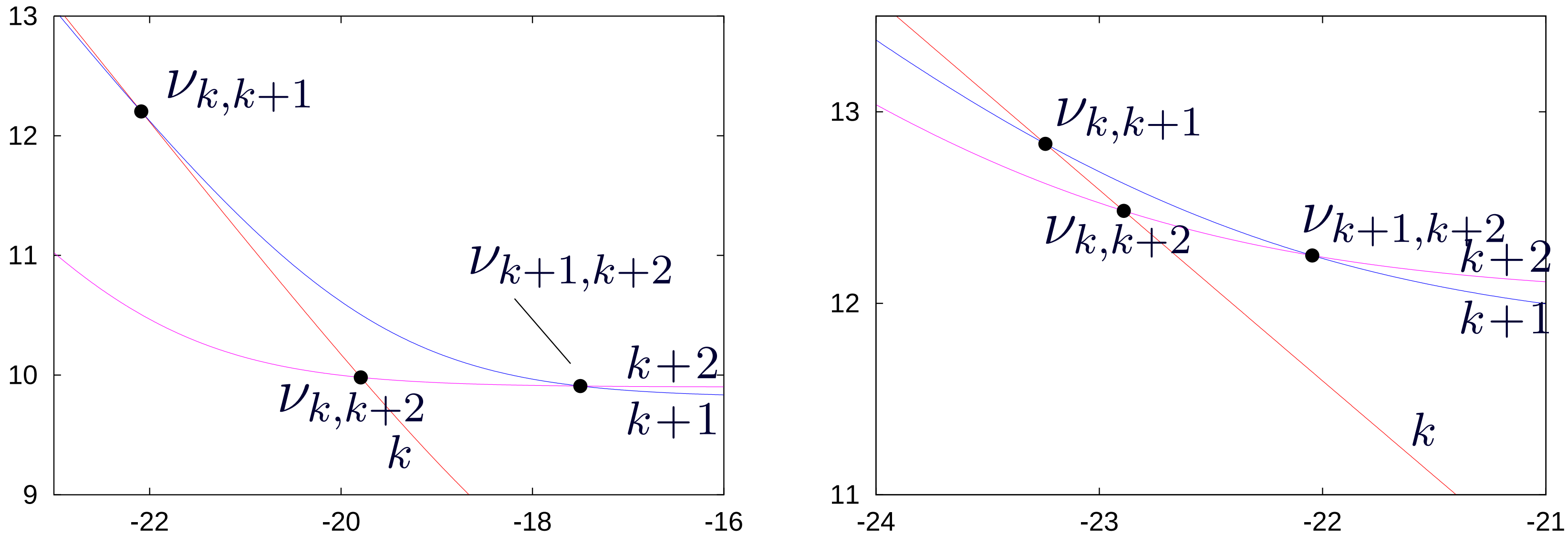
We are interested in minimizing $T(\nu, D_k)$ for a given ν. The optimal D_k depends on the arithmetic properties of γ.

Remark:

The frequencies γ_i, $i = 0, 1, 2$, verify $|p - q\gamma_i| \geq c/q^\tau$, $\tau \geq 1, \; c > 0$, and γ_3 satisfies $|p - q\gamma_3| \geq c/(q \log q)^\sigma$, $\sigma \geq 1$, $c \geq 0$ (this explains why the maxima in the plot increases like $\log \nu$).

Results on HBA for ν small (D_k large)

When $T(\nu, D_k) = T(\nu, D_l)$ a change of optimal from N_k/D_k to N_l/D_l, $l > k$, is produced. This gives $\nu_{k,l} = \frac{s_k - s_l}{D_l - D_k}$.



We display $\log(T(\nu, D_j))$, $j = k, k+1, k+2$, as a function of $\log(\nu)$. The $k+1$-th BA is hidden. Left: $\gamma = \gamma_2$. Right: $\gamma = \pi - 3$.

Thm. 1. **Two consecutive** harmonics associated to BA **cannot** be hidden.

2. If the $k+1$-th hamonic associated to BA is hidden then $q_{k+2} = 1$.

”Typical” measure-theoretical properties

Properties related to the CFE that hold for numbers in a set of full measure:

- The geometric mean of CFE quotients tends to the Kinchin constant KC ≈ 2.685452.
- Let D_n the BA denominators. Then $\lim_{n\to\infty} \log(D_n)/n \to$ LC $= \pi^2/(12\log(2))$ Levy constant.
- The Gauss map $x \to 1/x - [1/x]$ is ergodic and the probability of having k as a quotient is given by the Gauss-Kuzmin law: $P(k) = \log_2(1 + 1/(k^2 + 2k))$. For a “typical” number, its CFE is a sequence of realizations of **not independent** iid random variables.

Numerical checks (based on the first $\approx 5 \times 10^7$ first quotients) support that

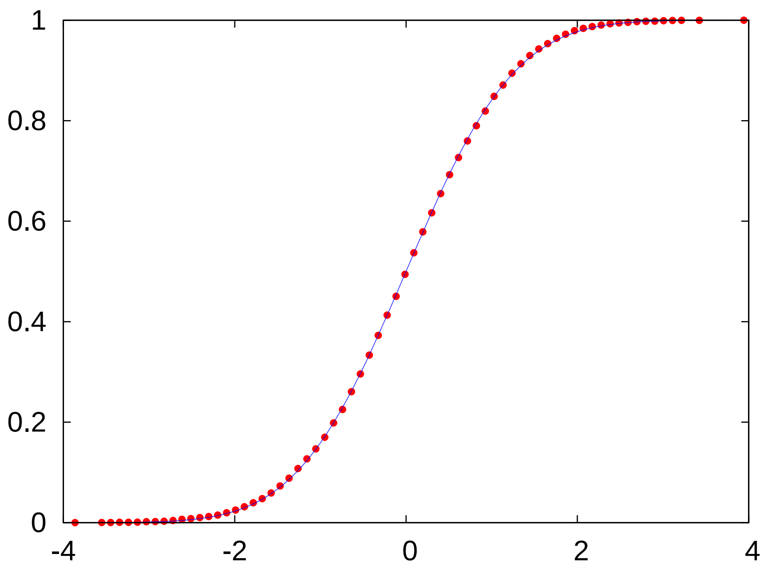
$$\gamma = \pi - 3,\ e^{\gamma_0},\ e^{\sqrt{2}} - 4,\ e^{\sqrt{3}} - 5, e^{\sqrt{5}} - 9,\ \text{and } e^{\sqrt{7}} - 14,$$

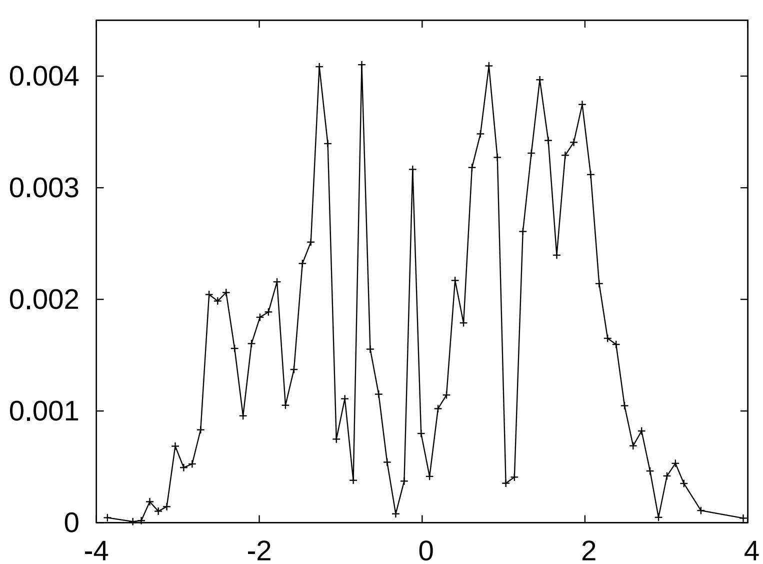
verify the previous “typical” properties.

A conjecture on the distribution of HBA

Conjecture: Under the assumptions on the homoclinic and the perturbation stated, for a set of ratios of two frequencies $(1, \gamma)$ of full measure, the distribution of HBA follows a normal law.

Numerical results for the system considered (we show results for $\gamma = \pi - 3$).





Counting the HBA in blocks of 1000 consecutive BA, we obtain that the CDF is $N(\mu, \sigma)$ with $\mu \approx 279.118$ and $\sigma \approx 9.604$ in all cases. That is, for our system and for a “typical” frequency γ we expect that more than one fourth of the BA are HBA. E.g. $\gamma = \pi - 3$: 2785810 HBA from the first 10^7 quotients.

What remains?

1. To theoretically justify the first-order Melnikov approach, and explain the very good agreement between the **symbolical** and **numerical** results.
2. To use the results on the splitting to derive a 4D (adapted) separatrix map (requires the passage time close to the complex-saddle point). Analyze the geometry of the phase space and the diffusive properties.
3. To carry out the study of the splitting for the 4D symplectic map case (rational/irrational **Krein collision** of eigenvalues).
4. ...

Thanks for your attention!!